Abstract

Tight sets in finite geometry

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Let Γ be a strongly regular graph with parameters (n, k, λ, μ). Let A be its adjacency matrix. if $0 < k < n-1$, then it is well known that:

- the matrix A has three eigenvalues k, e^+ and e^-;
- the eigenvalue k has multiplicity 1 and its eigenspace is generated by the all-one vector **j**;
- let V^+, V^- respectively, be the eigenspace corresponding to the eigenvalue e^+, e^- respectively, then $\mathbb{C}^n = \langle \mathbf{j} \rangle \perp V^+ \perp V^-$.

A vector $\chi \in \mathbb{C}^n$ is called a *weighted tight set* if $\chi \in \langle \mathbf{j} \rangle \perp V^+$, in other words, χ is orthogonal to V^-.

Strongly regular graphs occur frequently in finite geometry. Consider e.g. a finite classical polar space $\mathcal{P}$, and call Γ the graph with the points of $\mathcal{P}$ as vertices and two different vertices being adjacent if and only if the corresponding points are collinear. The graph Γ will be strongly regular and its parameters are well known. Let χ be a weighted tight set of Γ, then geometrically, χ associates a complex weight to each point of $\mathcal{P}$. When χ is a $0,1$ vector, the corresponding point set *behaves combinatorially as a disjoint union of generators*, and this property is often used as definition of a *tight set of a finite classical polar space*, probably for the first time by S.E. Payne in 1987.

A well studied example of tight sets of a particular polar space are the so-called Cameron-Liebler line classes. A *Cameron-Liebler line class with parameter x* is a set $\mathcal{L}$ of lines of $\mathrm{PG}(3,q)$ that has exactly x lines in common with any spread of $\mathrm{PG}(3,q)$. Using the Klein correspondence, it is clear that such an object is a tight set of the polar space $\mathrm{Q}^+(5,q)$. Cameron-Liebler line classes were introduced by Cameron and Liebler in 1982, and they conjectured (roughly spoken), that no non-trivial Cameron-Liebler exists, a conjecture that was disproven by the construction of an infinite family of Cameron-Liebler line classes with parameter $x = \frac{q^2+1}{2}$ by Bruen and Drudge in 1999.

In the talk, the history of tight sets in finite geometry will be surveyed. Their relation with strongly regular graphs will be play an important role. Then we focus and survey on Cameron-Liebler line classes, and non-exsistence results, and we present the construction of an infinite family with parameter $\frac{q^2-1}{2}$ for $q \equiv 5, 9 \pmod{12}$, which is joint work with J. Demeyer, K. Metsch, and M. Rodgers.

In the second part, we focus on ongoing research (jointly with J. Bamberg and F. Ihringer) on tight sets of finite classical polar spaces and their interaction with *ovoids*. An ovoid of a polar space is a $0,1$-vector $\chi \in \langle \mathbf{j} \rangle \perp V^-$. The combinatorial interaction between ovoids and tight sets is well understood, and the objective of this ongoing research is to obtain a unified approach to show non-existence of ovoids in several particular cases were this is known (or expected to be true) in a unified way.

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