Group rings of finite strongly monomial groups: central units and primitive idempotents

joint work with E. Jespers, G. Olteanu and Á. del Río

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Recend Trends in Rings and Algebras,
Murcia, June 3-7, 2013
Outline

Background information on units

Virtual basis of $\mathbb{Z}(\mathbb{U}(\mathbb{Z}G))$

Matrix units of each simple component in the rational group algebra $\mathbb{Q}G$

Description of subgroup of finite index in $\mathbb{U}(\mathbb{Z}G)$ for a class of metacyclic groups
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Group rings

Let $G$ be a group and $R$ a ring.
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$$\sum_{g \in G} r_g g,$$

with $r_g \in R$ and $r_g \neq 0$ for only finitely many coefficients.
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Operations:

$$\left( \sum_{g \in G} r_g g \right) + \left( \sum_{g \in G} s_g g \right) = \sum_{g \in G} (r_g + s_g) g$$

$$\left( \sum_{g \in G} r_g g \right) \cdot \left( \sum_{g \in G} s_g g \right) = \sum_{g, h \in G} r_g s_h g h$$
Let $R$ be a ring with unity 1.
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**Example**

Let $G$ be a finite group.

$$\mathcal{U}(\mathbb{Z}G) = ?$$
Example: Bass units

Let $G$ be a group, $g$ an element of $G$ of order $n$ and $k$ and $m$ positive integers such that $k^m \equiv 1 \mod n$. The **Bass (cyclic) unit** with parameters $g, k, m$ is the element in $\mathbb{Z}G$

$$u_{k,m}(g) = (1 + g + \cdots + g^{k-1})^m + \frac{(1 - k^m)}{n}(1 + g + g^2 + \cdots + g^{n-1})$$

with inverse in $\mathbb{Z}G$

$$(1 + g^k + \cdots + g^{k(i-1)})^m + \frac{(1 - i^m)}{n}(1 + g + g^2 + \cdots + g^{n-1})$$

where $i$ is any integer such that $ki \equiv 1 \mod n$. 
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**Theorem [Bass-Milnor]**

If $G$ is a finite abelian group, then the Bass units generate a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$. 
The **Wedderburn decomposition** of $\mathbb{Q}G$ is the decomposition into simple algebras

$$A_1 \oplus \cdots \oplus A_k.$$
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Each simple component is determined by a **primitive central idempotent** $e_i$, i.e.

$$A_i = \mathbb{Q}Ge_i.$$
Strong Shoda pairs

If $K \triangleleft H \leq G$ and $K \neq H$ then

$$\varepsilon(H, K) = \prod (\hat{K} - \hat{M}) = \hat{K} \prod (1 - \hat{M}),$$

where $M$ runs through the set of all minimal normal subgroups of $H$ containing $K$ properly.
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We set $\varepsilon(H, H) = \hat{H}.$
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A **strong Shoda pair** of $G$ is a pair $(H, K)$ of subgroups of $G$ such that

- $K \leq H \leq N_G(K)$
- $H/K$ is cyclic and a maximal abelian subgroup of $N_G(K)/K$
- the different $G$-conjugates of $\varepsilon(H, K)$ are orthogonal.
Strongly monomial groups

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Let \( \chi \) be an irreducible (complex) character of \( G \). \( \chi \) is **strongly monomial** if there is a strong Shoda pair \((H, K)\) of \( G \) and a linear character \( \theta \) of \( H \) with kernel \( K \) such that \( \chi = \theta^G \).

The group \( G \) is strongly monomial if every irreducible character of \( G \) is strongly monomial.

**Example**

Abelian-by-supersolvable groups are strongly monomial.
The Wedderburn decomposition for strongly monomial groups

**Theorem [Olivieri-del Río-Simón]**

Let $G$ be a finite strongly monomial group, then the Wedderburn decomposition is as follows:

$$
\mathbb{Q}G = \bigoplus_{(H,K)} \mathbb{Q}Ge(G, H, K) = \bigoplus_{(H,K)} M_n(\mathbb{Q}(\xi_k) \ast N_G(K)/H)
$$

with $(H, K)$ running on strong Shoda pairs of $G$. 
Virtual basis

It is well known that

\[ \mathbb{Z}(U(\mathbb{Z}G)) = \pm \mathbb{Z}(G) \times T, \]

where \( T \) is a finitely generated free abelian group.
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where \( T \) is a finitely generated free abelian group.

A virtual basis of \( \mathcal{Z}(U(\mathbb{Z}G)) \) is a set of multiplicatively independent elements of \( \mathcal{Z}(U(\mathbb{Z}G)) \) which generate a subgroup of finite index in \( \mathcal{Z}(U(\mathbb{Z}G)) \).
Strategy: We know how many elements a virtual basis of $\mathcal{Z}(U(\mathbb{Z}G))$ should have.
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**Theorem [Jespers-del Río-Olteanu-VG]**

Let $G$ be a finite strongly monomial group. Then the rank of $\mathcal{Z}(U(\mathbb{Z}G))$ equals

$$\sum_{(H,K)} \left( \frac{\varphi([H : K])}{k(H,K)[N : H]} - 1 \right),$$

where $(H, K)$ runs through a complete and non-redundant set of strong Shoda pairs of $G$, $h$ is such that $H = \langle h, K \rangle$ and

$$k(H,K) = \begin{cases} 
1, & \text{if } hh^n \in K \text{ for some } n \in N_G(K); \\
2, & \text{otherwise.}
\end{cases}$$
Main strategy

Take an arbitrary central unit $u$ in $\mathcal{Z}(U(\mathbb{Z}G)) \subset \mathbb{Q}G$. 

Strategy

1. Compute a basis of $U(\mathbb{Z}[\zeta[H : K]] N_G(K)/H)$
2. Compute units in $\mathbb{Z}G$ projecting to the basis in $U(\mathbb{Z}[\zeta[H : K]] N_G(K)/H)$ and trivially in the other components.
Main strategy

Take an arbitrary central unit $u$ in $\mathbb{Z}(\mathcal{U}(\mathbb{Z}G)) \subset \mathbb{Q}G$. We can write $u$ as follows

$$u = \sum_{(H,K)} ue(G, H, K) = \prod_{(H,K)} (1 - e(G, H, K) + ue(G, H, K)).$$
Virtual basis of the center

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Take an arbitrary central unit $u$ in $\mathcal{Z}(U(\mathbb{Z}G)) \subset \mathbb{Q}G$. We can write $u$ as follows

$$u = \sum_{(H,K)} u_e(G,H,K) = \prod_{(H,K)} (1 - e(G,H,K) + u_e(G,H,K)).$$

Hence it is necessary and sufficient to construct a virtual basis in each

$$\mathcal{Z}(\mathbb{Z}Ge(G,H,K) + \mathbb{Z}(1 - e(G,H,K))) \approx \mathbb{Z}[\zeta_{[H:K]}]^{N_G(K)/H} + \mathbb{Z}(1 - e(G,H,K)).$$
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$$\mathbb{Z}(\mathbb{Z}G e(G, H, K) + \mathbb{Z}(1 - e(G, H, K))) \approx \mathbb{Z}[^{\zeta[H:K]}_N G(K)/H] + \mathbb{Z}(1 - e(G, H, K)).$$

Strategy

1. Compute a basis of $U(\mathbb{Z}[^{\zeta[H:K]}_N G(K)/H])$
2. Compute units in $\mathbb{Z}G$ projecting to the basis in $U(\mathbb{Z}[^{\zeta[H:K]}_N G(K)/H])$ and trivially in the other components.
Step 1: Cyclotomic units

If $n > 1$ and $k$ is an integer coprime with $n$ then

$$\eta_k(\zeta_n) = \frac{1 - \zeta_n^k}{1 - \zeta_n} = 1 + \zeta_n + \zeta_n^2 + \cdots + \zeta_n^{k-1}$$

is a unit of $\mathbb{Z}[\zeta_n]$. These units are called the \textbf{cyclotomic units} of $\mathbb{Q}(\zeta_n)$. 
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**Theorem [Washington]**

\[
\{ \eta_k(\zeta_{p^n}) \mid 1 < k < \frac{p^n}{2}, \ p \nmid k \} \text{ generates a free abelian subgroup of finite index in } U(\mathbb{Z}[\zeta_{p^n}]) \text{ when } p \text{ is prime.}
\]
Step 1: Cyclotomic units

For a subgroup $A$ of $\text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ and $u \in \mathbb{Q}(\zeta_{p^n})$, we define $\pi_A(u)$ to be $\prod_{\sigma \in A} \sigma(u)$. 
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Lemma [Jespers-del Río-Olteanu-VG]

Let $A$ be a subgroup of $\text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$. Let $I$ be a set of coset representatives of $\mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$ modulo $\langle A, \phi^{-1} \rangle$ containing 1. Then the set

$$\{ \pi_A (\eta_k(\zeta_{p^n})) \mid k \in I \setminus \{1\} \}$$

is a virtual basis of $\mathcal{U}(\mathbb{Z}[\zeta_{p^n}]^A)$.
Step 2: Generalized Bass units

If $G$ is a finite group, $M$ a normal subgroup of $G$, $g \in G$ and $k$ and $m$ positive integers such that $\gcd(k, |g|) = 1$ and $k^m \equiv 1 \mod |g|$. 
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$$u_{k,m}(1 - \hat{M} + g \hat{M}) = 1 - \hat{M} + u_{k,m}(g)\hat{M}.$$

Observe that any element \( b = u_{k,m}(1 - \hat{M} + g \hat{M}) \) is an invertible element of \( \mathbb{Z}G(1 - \hat{M}) + \mathbb{Z}G\hat{M} \). As this is an order in \( \mathbb{Q}G \), there is a positive integer \( n \) such that \( b^n \in \mathcal{U}(\mathbb{Z}G) \).
If $G$ is a finite group, $M$ a normal subgroup of $G$, $g \in G$ and $k$ and $m$ positive integers such that $\gcd(k, |g|) = 1$ and $k^m \equiv 1 \mod |g|$. Then we have

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Observe that any element $b = u_{k,m}(1 - \hat{M} + g\hat{M})$ is an invertible element of $\mathbb{Z}G(1 - \hat{M}) + \mathbb{Z}G\hat{M}$. As this is an order in $\mathbb{Q}G$, there is a positive integer $n$ such that $b^n \in \mathcal{U}(\mathbb{Z}G)$.

Let $n_{G,M}$ denote the minimal positive integer satisfying this condition for all $g \in G$. Then we call the element

$$u_{k,m}(1 - \hat{M} + g\hat{M})^{n_{G,M}} = u_{k,mn_{G,M}}(1 - \hat{M} + g\hat{M})$$

a **generalized Bass unit** based on $g$ and $M$ with parameters $k$ and $m$.  

**Step 2: Generalized Bass units**
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Let $k$ be a positive integer coprime with $p$ and let $r$ be an arbitrary integer.
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Let \( k \) be a positive integer coprime with \( p \) and let \( r \) be an arbitrary integer. For every \( 0 \leq j \leq s \leq n \) we construct recursively the following products of generalized Bass units of \( \mathbb{Z}H \):

\[
c_{s}^{j}(H, K, k, r) = 1,
\]

and, for \( 0 \leq j \leq s - 1 \),

\[
c_{s}^{j}(H, K, k, r) = \prod_{h \in H} u^{k, O_{p}^{n}(k)_{n H, K}(g_{r p}^{n} - s h \hat{K} + 1 - \hat{K})}^{p s - j - 1} \prod_{l = j + 1}^{s - 1} c_{s}^{l}(H, K, k, r) - 1 \prod_{l = 0}^{s + l - j - 1} c_{s}^{l}(H, K, k, r) - 1.
\]
Step 2: Generalized Bass units

Let $k$ be a positive integer coprime with $p$ and let $r$ be an arbitrary integer. For every $0 \leq j \leq s \leq n$ we construct recursively the following products of generalized Bass units of $\mathbb{Z}H$:

$$c^s_j(H, K, k, r) = 1,$$

$$c^{s-1}_j\left(\prod_{l=j+1}^{s} c^s_l(H, K, k, r) - 1\right) - \left(\prod_{l=0}^{j} c^{s+l-j}(H, K, k, r) - 1\right).$$
Virtual basis of the center

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\[
c_s^s(H, K, k, r) = 1,
\]

and, for \( 0 \leq j \leq s - 1 \),

\[
c_j^s(H, K, k, r) = \left( \prod_{h \in H_j} u_{k, O_{p^n}(k)n_{H, K}}(g^{rp^n}hK + 1 - \hat{K}) \right)^{ps-j-1} \left( \prod_{l=j+1}^{s-1} c_l^s(H, K, k, r)^{-1} \right) \left( \prod_{l=0}^{j-1} c_l^{s+l-j}(H, K, k, r)^{-1} \right).
\]
Step 2: Generalized Bass units

Proposition [Jespers-del Río-Olteanu-VG]

Let $H$ be a finite group and $K$ a subgroup of $H$ such that $H/K = \langle gK \rangle$ is cyclic of order $p^n$. Let

$\mathcal{H} = \{ L \leq H \mid K \leq L \} = \{ H_j = \langle g^{p^n-j}, K \rangle \mid 0 \leq j \leq n \}$. Let $k$ be a positive integer coprime with $p$ and let $r$ be an arbitrary integer. Then

$$\rho_{H_{j_1}}(c_j^s(H, K, k, r)) = \begin{cases} 
\eta_k(\zeta_r^{s-j})O_{p^n}(k)p^{s-1}n_{H,K}, & \text{if } j = j_1; \\
1, & \text{if } j \neq j_1.
\end{cases} \quad (1)$$

for every $0 \leq j, j_1 \leq s \leq n$. 

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Main Theorem

Theorem [Jespers-del Río-Olteanu-VG]

Let $G$ be a strongly monomial group such that there is a complete and non-redundant set $S$ of strong Shoda pairs $(H, K)$ of $G$ with the property that each $[H : K]$ is a prime power. For every $(H, K) \in S$, let $T_K$ be a right transversal of $N_G(K)$ in $G$, let $I_{(H,K)}$ be a set of representatives of $\mathcal{U}(\mathbb{Z}/[H : K]\mathbb{Z})$ modulo $\langle N_G(K)/H, -1 \rangle$ containing 1 and let $[H : K] = p_{(H,K)}^{n_{(H,K)}}$, with $p_{(H,K)}$ prime. Then

$$\left\{ \prod_{t \in T_K} \prod_{x \in N_G(K)/H} c_0^{n_{(H,K)}}(H, K, k, x)^t : (H, K) \in S, k \in I_{(H,K)} \setminus \{1\} \right\}$$

is a virtual basis of $\mathbb{Z}(\mathcal{U}(\mathbb{Z}G))$. 
Virtual basis of the center

Examples of groups satisfying the conditions

- $C_{q^m} \rtimes C_{p^n}$ with $C_{p^n}$ acting faithfully on $C_{q^m}$
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- $C_q^m \rtimes C_p^n$ with $C_p^n$ acting faithfully on $C_q^m$
- $A_4$
- $D_{2n} = \langle a, b \mid a^n = b^2 = 1, \ a^b = a^{-1} \rangle \iff n$ is a power of a prime
Examples of groups satisfying the conditions

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- $A_4$
- $D_{2n} = \langle a, b \mid a^n = b^2 = 1, \ a^b = a^{-1} \rangle \iff n$ is a power of a prime
- $Q_{4n} = \langle x, y \mid x^{2n} = y^4 = 1, \ x^n = y^2, \ xy = x^{-1} \rangle \iff n$ is a power of 2
- ...

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Idea

Each simple component of $\mathbb{Q}G$ is isomorphic to a matrix algebra.
Each simple component of $\mathbb{Q}G$ is isomorphic to a matrix algebra. We want to know which group ring elements represent the matrix units $E_{ij}$:

$$E_{ij} = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\end{pmatrix}.$$
Idea

For a strongly monomial group $G$, each simple component of $\mathbb{Q}G$ is of the form

$$\mathbb{Q}Ge(G, H, K) = M_n(\mathbb{Q}He(H, K) \ast NG(K)/H)$$

for a strong Shoda pair $(H, K)$. 
Matrix units

Idea

For a strongly monomial group $G$, each simple component of $\mathbb{Q}G$ is of the form

$$\mathbb{Q}Ge(G, H, K) = M_n(\mathbb{Q}H\varepsilon(H, K) \ast N_G(K)/H)$$

for a strong Shoda pair $(H, K)$.

When the twisting of $\mathbb{Q}N_G(K)\varepsilon(H, K) \ast N_G(K)/H$ is trivial, Reiner provides an explicit isomorphism

$$\psi : \mathbb{Q}H\varepsilon(H, K) \ast N_G(K)/H \rightarrow M_{[N_G(K):H]}(F).$$
Theorem [Jespers-del Río-Olteanu-VG]

Let \((H, K)\) be a strong Shoda pair of a finite group \(G\) such that \(\tau(nH, n'H) = 1\) for all \(n, n' \in NG(K)\). Let \(P, A \in M_n(F)\) be the matrices

\[
P = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & -1 & 0 & \cdots & 0 & 0 \\
1 & 0 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & -1 & 0 \\
1 & 0 & 0 & \cdots & 0 & -1
\end{pmatrix}
\]

and

\[
A = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\]

Then

\[
\{ E_{x'x} := x' \hat{T}_1 \epsilon(H, K)x^{-1} \mid x, x' \in T_2 \langle x_e \rangle \}
\]

is a complete set of matrix units of \(\mathbb{Q}Ge(G, H, K)\) where

\(x_e = \psi^{-1}(PAP^{-1})\), \(T_1\) is a transversal of \(H\) in \(NG(K)\) and \(T_2\) is a right transversal of \(NG(K)\) in \(G\).
Examples of groups satisfying the conditions

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Main ingredients

Let $\mathcal{O}$ be an order in a division algebra $D$. For an ideal $Q$ of $\mathcal{O}$ we denote by $E(Q)$ the subgroup of $\text{SL}_n(\mathcal{O})$ generated by all $Q$-elementary matrices, that is $E(Q) = \langle I + qE_{ij} \mid q \in Q, \ 1 \leq i, j \leq n, \ i \neq j, \ E_{ij} \text{ a matrix unit} \rangle$.

**Theorem [Bass-Vaseršteǐn-Liehl-Venkataramana]**

If $n \geq 3$ then $[\text{SL}_n(\mathcal{O}) : E(Q)] < \infty$. If $\mathcal{U}(\mathcal{O})$ is infinite then $[\text{SL}_2(\mathcal{O}) : E(Q)] < \infty$. 
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**Theorem [Bass-Vaseršteǐn-Liehl-Venkatararamana]**

If $n \geq 3$ then $[\text{SL}_n(\mathcal{O}) : E(Q)] < \infty$. If $U(\mathcal{O})$ is infinite then $[\text{SL}_2(\mathcal{O}) : E(Q)] < \infty$.

Since $\text{GL}_n(\mathcal{O})$ is generated by $\text{SL}_n(\mathcal{O})$ and its center, our construction of central units and matrix units is sufficient to describe a subgroup of finite index in $U(\mathbb{Z}G)$ for some groups $G$. 
Theorem [Jespers-del Río-Olteanu-VG]

Let $G = C_{q^m} \rtimes C_{p^n}$ be a finite metacyclic group with $C_{p^n} = \langle b \rangle$ acting faithfully on $C_{q^m} = \langle a \rangle$ and with $p$ and $q$ different primes. Assume that either $q \neq 3$, or $n \neq 1$ or $p \neq 2$. Then the following two groups are finitely generated nilpotent subgroups of $\mathcal{U}(\mathbb{Z}G)$:

$$V_j^+ = \left\langle 1 + p^n t_j^2 yx_j^h b x_j^{-k} \mid y \in \overline{\langle a \rangle \langle b \rangle}, \ h, k \in \{1, \ldots, p^n\}, \ h < k \right\rangle,$$

$$V_j^- = \left\langle 1 + p^n t_j^2 yx_j^h b x_j^{-k} \mid y \in \overline{\langle a \rangle \langle b \rangle}, \ h, k \in \{1, \ldots, p^n\}, \ h > k \right\rangle.$$

Hence $V^+ = \prod_{j=1}^m V_j^+$ and $V^- = \prod_{j=1}^m V_j^-$ are nilpotent subgroups of $\mathcal{U}(\mathbb{Z}G)$. Furthermore, the group

$$\langle U, V^+, V^- \rangle,$$

with $U$ a virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, is of finite index in $\mathcal{U}(\mathbb{Z}G)$. 
Reference

- E. Jespers, G. Olteanu, Á. del Río, I. Van Gelder, Group rings of finite strongly monomial groups: Central units and primitive idempotents, Journal of Algebra, Volume 387, 1 August 2013, Pages 99-116