Software package for mosaic-Hankel structured low-rank approximation

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Abstract—This paper presents the SLRA package (http://slra.github.io)—C software with interface to MATLAB, Octave, and R for solving low-rank approximation problems with the following features: mosaic Hankel structured approximating matrix, weighted 2-norm approximation criterion, and fixed and missing elements in the approximating matrix. The package has applications in system identification, machine learning, and computer algebra. The paper gives an overview of the features of the package, including the wrapper functions for system identification (IDENT package) and approximate greatest common divisor computations. An addendum to the paper, available from http://homepages.vub.ac.be/~imarkovs/slra-demo includes examples that demonstrate the usage, versatility, and efficiency of the software.

II. PROBLEM FORMULATION

The SLRA package solves problem (SLRA). Typical code is

```
minimize over \( \hat{p} \) \( \| p - \hat{p} \|_W \)
subject to \( \text{rank}(S(\hat{p})) \leq r. \)
```

Due to the rank constraint, problem (SLRA) is nonconvex. Except for a few notable exceptions (unstructured approximation in the spectral or Frobenius norm, rank-1 weight matrix \( W \), and circulant structure) there are no polynomial time global solution methods for (SLRA). State-of-the-art methods split into convex relaxation and local optimization heuristics. In the special case of (generalized) Hankel and Toeplitz structured matrices, the class of subspace methods [4], [5], [6] provides also an effective and efficient heuristic for structured low-rank approximation.

This paper presents a local optimization-based software package for mosaic Hankel structured low-rank approximation, hosted at https://github.com/slra/slra/

The SLRA package is built upon an implementation of the following methods for structured low-rank approximation: (i) variable projection (kernel representation) [2], [7], [8], [9] and (ii) penalisation (image representation) [10]. In the variable projection approach, the problem is recast as optimization on a Grassmann manifold [11]. The package includes wrapper functions for (a) system identification (the IDENT package) described in [12] and (b) approximate greatest common divisor (AGCD) computations [13].

We give a comprehensive overview of the features implemented in the package. The previous publication [7] is outdated, as it takes into account only the methods of [7], [8], [9]. Here, we describe, in addition, the methods of [9], [10], [11], [12], [13]. Thus, the paper gathers the description of the parameters and options of the methods scattered in the literature.

The paper is organized as follows. Section II defines the problem solved by the package. Section III describes the solvers included in the package and their options. Section IV presents a MATLAB wrapper function for using the SLRA package for linear time-invariant system identification. Section V describes the wrapper functions for AGCD computations. An addendum to the paper, available from http://homepages.vub.ac.be/~imarkovs/slra-demo, contains numerical examples of using the software.

II. PROBLEM FORMULATION

The SLRA package solves problem (SLRA). Typical code is
[ph, info] = slra(p, s, r, opt)
where p is the data vector, s encodes the structure and the weighted norm, and r is the rank of the approximation.

In the following subsection, we give more details on the structure specification and the weighted 2-norm (contained in the parameter s). The parameter opt determines the solver being used and contains the parameters of the solvers (see Section III for more details).

The output parameter ph contains the computed approximation \( \hat{p} \). The structure info contains additional output information from the optimization, such as:

- info.fmin: the value of the cost function \( \| p - \hat{p} \|_W^2 \);
- info.iter: number of iterations;
- info.time: execution time;
- info.Rh: low-rank certificate—a full row rank matrix \( \hat{R} \in \mathbb{R}^{(m-r) \times m} \), such that
  \[
  \hat{R} \hat{S}(\hat{p}) = 0. \tag{KER}
  \]

Additional information may be returned in info for some of the solvers.

A. Structure specification

The SLRA package deals with affine structures, i.e., structured matrices that form an affine subset of \( \mathbb{R}^{m \times n} \).

The package allows for the following matrix structures:

- **General affine structures**:
  \[
  S(p) = \Phi(S_0 + \sum_{k=1}^{s.w} p_k S_k), \tag{S}
  \]
  where \( S_k \in \mathbb{R}^{m' \times n} \) are basis matrices and \( \Phi \in \mathbb{R}^{m \times m'} \) is a full row rank matrix (an identity matrix by default). Moreover, \( S_k \) must be orthogonal (i.e., \( \langle S_k, S_j \rangle = \text{trace}(S_k S_j^\top) = 0 \) for \( k \neq j \)) and consist only of zeros and ones. The latter condition ensures that the linear part (i.e., \( \sum_{k=1}^{s.w} p_k S_k \)) contains in each entry either zero, or an element of the vector \( p \), so it can be compactly represented as an integer matrix of the indices of the elements of \( p \).

For example, the matrix structure
  \[
  S(p) = \begin{bmatrix}
    4 & 5 & 6 & p_1 \\
    5 & 6 & p_1 & p_2 \\
    6 & p_1 & p_2 & p_3
  \end{bmatrix}
  \]
  can be specified as follows:
  \[
  \begin{align*}
  s.s0 &= [4 \ 5 \ 6 \ 0; \ldots
    \ 5 \ 6 \ 0 \ 0; \ldots
    \ 6 \ 0 \ 0 \ 0];
  s.ts & = [0 \ 0 \ 0; \ldots
    \ 0 \ 0 \ 1; \ldots
    \ 0 \ 1 \ 2 \ 3];
  \end{align*}
  \]
  An example for the matrix \( \Phi \) will be given next.

- **Mosaic Hankel structures**

  The mosaic Hankel \[ \mathcal{H}_{m,n}(p) \]
  is a generalization of the classical Hankel structure
  \[
  \mathcal{H}_{m,n}(p) := \begin{bmatrix}
    p_1 & p_2 & p_3 & \cdots & p_n
    p_2 & p_3 & \cdots & p_{n+1}
    p_3 & \cdots & \vdots & \vdots
    \vdots & \vdots & \ddots & \vdots
    p_m & p_{m+1} & \cdots & p_{m+n-1}
  \end{bmatrix}
  \]
  A mosaic Hankel matrix \( \mathcal{H}_{m,n}(p) \) is a block matrix with scalar Hankel blocks. The vectors
  \[
  \mathbf{m} := [m_1 \ \cdots \ m_q] \quad \text{and} \quad \mathbf{n} := [n_1 \ \cdots \ n_N]
  \]
  define the sizes of the scalar Hankel blocks and therefore uniquely specify the structure. The parameter vector \( p \in \mathbb{R}^{n_p} \) is the concatenation of chunks \( p^{(i,j)} \)
  \[
  p = (p^{(1,1)}, \ldots, p^{(q,1)}, \ldots, p^{(1,N)}, \ldots, p^{(q,N)}),
  \]
  where \( p^{(i,j)} \in \mathbb{R}^{m_i+n_j-1} \).

  The software package supports mosaic-Hankel-like structures of the form
  \[
  \mathcal{S}(p) := \Phi \mathcal{H}_{m,n}(p), \quad (\Phi \mathcal{H}_{m,n})
  \]
  where \( \Phi \) is a full row rank matrix, further extending the class of mosaic Hankel matrices to (mosaic) Hankel-like matrices.

  For example, a Toeplitz matrix
  \[
  \begin{bmatrix}
    p_3 & p_4 & p_5 \\
    p_2 & p_3 & p_4 \\
    p_1 & p_2 & p_3
  \end{bmatrix}
  \]
  can be specified as follows:
  \[
  \begin{align*}
  s.m &= 3; \ s.n = 3; \ s.Phi = [0 \ 0 \ 1; \ldots
    \ 0 \ 1 \ 0; \ldots
    \ 1 \ 0 \ 0];
  \end{align*}
  \]

  A Toeplitz-plus-Hankel matrix can be defined in a similar way, as shown in [7].

B. Approximation criterion

The approximation criteria supported by the package are weighted 2-norms \( \| \cdot \|_W \), with two types of \( W \):

- A diagonal weight matrix \( W = \text{diag}(w) \), where \( w \) is a vector with nonnegative real elements from 0 to \( \infty \). Weight \( w_i = 0 \) corresponds to treating \( p_i \) as a missing value. In the other extreme \( w_i = \infty \), \( \hat{p}_i \) should match the data \( p_i \) (an equality constraint in the optimization problem (SLRA)).

  In general, the weights are specified by a given vector \( s.w \) of length \( n_p \). For mosaic-Hankel-like structure, block-wise weights can be specified in two ways:

  - if \( s.w \) is of length \( qN \), then the values are the weights for the whole chunks \( p^{(i,j)} \) in the vectorisation order;
  - if \( s.w \) is of length \( q \), the value of the \( i \)-th element of \( s.w \) is a weight for \( p^{(i,j)} \) for every \( j \).

  Another option is to use a general symmetric positive semi-definite matrix \( W \in \mathbb{R}^{n_p \times n_p} \).
III. SLRA PACKAGE: SOLVERS AND OPTIONS

This section gives an overview of methods implemented in the SLRA package. The package contains:

1) fast C++ implementation of the variable projection (VARPRO) method for mosaic Hankel matrices [9];
2) an implementation of the VARPRO method for SLRA with missing data [8]; this method is also called "experimental Matlab solver" in [7].
3) the factorization approach to SLRA based on a penalty method [10].

In Table I the supported structure and weight specification for each methods are listed.

### Table I

<table>
<thead>
<tr>
<th>opt.solver</th>
<th>1)</th>
<th>2)</th>
<th>3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>General affine structure</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Mosaic-Hankel-like (ΦH_{m,n})</td>
<td>+</td>
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<td>+</td>
</tr>
<tr>
<td>elementwise weights w_k ∈ [0, ∞)</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>missing data w_k = 0</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>semi-definite weight matrix W</td>
<td>-</td>
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By default, the efficient VARPRO method is used, if a certain structure/weight specification is not supported, the method 2) is used. The user can select the method by using the field opt.solver.

A. Variable projection methods

In the variable projection [2] approach, the rank constraint is replaced by the equivalent constraint (KER) and problem (SLRA) is restated as an equivalent bi-level optimization problem

\[ \text{minimize over } R \in \mathbb{R}^{d \times m} \quad f(R) \]

subject to \( R \) has full row rank,

\[ f(R) := \min_p ||p - \bar{p}||_W^2 \text{ subject to } RS(\bar{p}) = 0, \]

where \( d = m - r \) and \( f(R) \) is a solution of the (SLRA) problem for fixed kernel matrix \( R \).

The idea of the VARPRO method is that the cost function \( f(R) \) and its derivatives can be found in a closed form, and the cost function can be minimized using conventional optimization methods, see [2], [8].

1) Search space and constraints on the kernel: The cost function has an invariance property

\[ f(R) = f(U R), \text{ for any nonsingular } U \in \mathbb{R}^{d \times d}. \]

Hence, the search space is the Grassmann manifold [11]. In addition, the package allows for linear constraints on the kernel. Two types of constraints can be used:

1) General linear constraint

\[ R = R'(\theta) := \text{vec}_d^{-1}(\theta \Psi), \text{ where } \theta \in \mathbb{R}^{n_{\theta}}, \]

and \( \Psi \in \mathbb{R}^{n_{\theta} \times m} \), i.e., it is assumed that \( R \) belongs to a linear subspace of \( \mathbb{R}^{d \times m} \).

2) Matrix-product constraint

\[ R = R(\theta) := \Theta \Psi, \text{ where } \Theta \in \mathbb{R}^{d \times m''}, \]

and \( \Psi \in \mathbb{R}^{m'' \times m} \) is a full row rank matrix.

The matrix-product linear constraint is a special case of the general linear constraint since

\[ \text{vec}^T(\Theta \Psi) = \text{vec}^T(\Theta)(\Psi \otimes I_d). \]

Note 1: Both linear constraints are supported by both variable projection methods (efficient C++ solver and experimental MATLAB solver). Note that in [7] it was written that only the MATLAB solver supports the general linear constraint.

Both linear constraints are handled by giving the field opt.psi. The package determines the constraint under consideration based on the size of the matrix \( \Psi \). By default, it is assumed that \( \Psi = I_m \).

B. Efficient C++ solver and its options

The efficient C++ solver (opt.solver = 'c') implements the method described in [9]. It can handle mosaic Hankel structures (ΦH_{m,n}) and elementwise positive weights, with a linear complexity in the size of data evaluation complexity of cost function and its derivatives (assuming \( m \) and \( d \) fixed). General structure (S), missing data, or general weight matrices cannot be handled. The solver can use optimization algorithms implemented in the GSL (GNU Scientific Library) library [15] and own implementation of Levenberg-Marquardt method.

1) GSL and constraints: The methods implemented in the GSL library are not designed for optimization on manifolds, hence these methods work under two assumptions:

1) matrix-product constraint;
2) and additional constraint

\[ \Theta = [X - I_d], \text{ where } X \in \mathbb{R}^{d \times (m'' - d)}. \]

Note 2: The latter assumption parametrizes the Grassmann manifold minus a subset of measure zero. The whole Grassmann manifold can be considered by employing permutations (in \( \Psi \)), as explained in [11].

The employed methods are:

- opt.method = 'n...': derivative-free (Nelder-Mead) methods;
- opt.method = 'q...': quasi-Newton methods;
- opt.method = 'l...': Levenberg-Marquardt (see the next subsection).

The second letter of the string opt.method determines the sub-method being used. The possible sub-methods can be found in the documentation of the package, see, cpp/OptimizationOptions.h.

2) Levenberg-Marquardt methods: The Levenberg-Marquardt methods use sum of squares representation of the cost function

\[ f(R) = ||g(R)||_2^2, \]

7167
where \( g: \mathbb{R}^{d \times m} \to \mathbb{R}^{n_g} \) is a vector-valued map. By default, the Cholesky factorisation is used as \( g \) (the function \( g_s \) in [9, Alg. 6]). Another option (\texttt{opt.ls correction = 1}) is to use the function \( g = \Delta p' (R) \) (see [9, p. 435]).

The LM method from the GSL library (\texttt{opt.method = 'lm'}) is applicable only to the matrix-product constraint on the kernel. The general linear constraint can be handled by an own implementation of the 'data-driven local coordinates' modification of the Levenberg-Marquardt method, described in [11, §3.2]. This method (chosen by \texttt{opt.method = 'lm'}) automatically takes care of the linear dependencies imposed by the Grassmann manifold and of the general or matrix-product linear constraint.

Note 3: For \texttt{opt.method = 'p...'}, in the case of a matrix-product constraint, still the additional constraint \( (X) \) is used. If the constraint \( (X) \) needs to be avoided, one should specify \texttt{opt.avoid.xi = 1}.

3) Additional parameters: The available stopping criteria are:

- \texttt{opt.maxiter}: maximum number of iterations;
- \texttt{opt.epsgrad}: lower bound on \( \|\nabla f\| \);
- \texttt{opt.epsabs, opt.epsrel}: stops if
  \[ \|X_i - X_{i+1}\| < \text{epsabs} + \text{epsrel} \cdot \|X_i\|. \]
- \texttt{opt.maxx}: when parameterisation \( (X) \) is used, stops if \( \|X\|_\infty \) exceeds \texttt{opt.maxx}.

C. Experimental (MATLAB) solver

The MATLAB solver (\texttt{opt.solver = 'm'}) based on the results of [8], uses general purpose optimisation routines such as \texttt{fmincon/fminunc}.

The constraint on \( R \) is imposed in one of the following ways:

- hard constraint (default): \( RR^T = I_d \). In that case, \texttt{fmincon} is used. Note that, in this case, the cost function is minimised on the Stiefel manifold.
- ‘penalty’ term (if \texttt{opt.method = 'reg'}):
  \[ f(R) + \gamma \|RR^T - I_d\|^2_2, \]
  together with the \texttt{fminunc}. In fact, as shown in [8], this penalty term is exact (enforces the hard constraint \( RR^T = I_d \)).

Note that:

- If the linear constraints on the kernel are present, they are substituted in \( RR^T = I_d \).
- Standard optimisation options (such as \texttt{opt.maxiter, opt.tolx, etc.}) can be passed in the structure \texttt{opt}.

D. Factorization (alternating least squares) methods

The method \texttt{opt.solver = 'r'} is an implementation of the method in [10]. (SLRA) is reformulated as

\[
\min_{p \in \mathbb{R}^{m \times r}, L \in \mathbb{R}^{r \times n}} \|p - \Pi_S \text{vec}(PL)\|^2_W + \lambda_k \|PL - P_S(PL)\|^2_F, \tag{2}
\]

where \( \lambda_k \) is the penalty parameter controlling how close the approximation is to the set of structured matrices.

The method proceeds by alternately minimising the cost function for fixed \( P \) and \( L \). The hyper parameter \( \lambda_k \) depends on the iteration and is gradually increasing. The additional input parameters for this method are:

- \texttt{opt.P_init}: initial approximation for \( P \) (default: based on the SVD);
- \texttt{opt.lambda_init}: initial value of \( \lambda_k \);
- \texttt{opt.lambda_max}: maximal value of \( \lambda_k \);
- \texttt{opt.max_inner_iter}: maximal number of inner iterations (the maximal number of iterations when \( \lambda_k \) does not change);

Additional output parameters include:

- \texttt{info.P, info.L}: computed \( P \) and \( L \) matrices;
- \texttt{info.lambda: last value of lambda;}
- \texttt{info.outer_iterations: number of of outer iterations (changes of \( \lambda_k \));}
- \texttt{info.inner_iterations: number of inner iterations for each outer iteration;}

IV. WRAPPER FUNCTIONS FOR SYSTEM IDENTIFICATION

This section describes the functions of the IDENT package [12] for linear time-invariant system identification. System identification problems can be solved via a structured low-rank approximation, but a wrapper function is needed in order to convert the system identification problem to the input representation for the SLRA solver.

A. The identification problem

We use the behavioral language [16], where a dynamical system is viewed as a collection of trajectories. Let \( \mathcal{L}^q_n \) be the model class of linear-time invariant systems with \( q \) manifest variables (inputs and outputs) of bounded complexity (at most \( n \) inputs and lag at most \( \ell \)). Given a system \( \mathcal{B} \in \mathcal{L}^q_n \), time series \( w_d = \{w_d^1, \ldots, w_d^N\} \), and weights \( v = \{v^1, \ldots, v^N\} \), the lack of fit (misfit) between the model \( \mathcal{B} \) and the data \( w_d \) is defined as

\[
M(w_d, \mathcal{B}) := \min_{\hat{w}^1, \ldots, \hat{w}^N \in \mathcal{B}} \sum_{k=1}^{N} \|w_k^d - \hat{w}^k\|^2_v,
\]

Note that \( \hat{w}^v \) is the projection of \( w_d^v \) on \( \mathcal{B} \).

The identification problem solved by the IDENT package is: given a (set of) time series \( w_d \), complexity specification \( (m, \ell) \), and weights \( v \), find the model that minimizes the misfit criterion over all models with bounded complexity:

\[
\mathcal{B} := \arg \min_{\mathcal{B} \in \mathcal{L}^q_n} M(w_d, \mathcal{B}). \tag{SYSID}
\]

As shown in [1, Section 3.1] (SYSID) yields the maximum likelihood estimator in the errors-in-variables setting [17]

\[
w_k^d = \hat{w}^k + \hat{w}^k,
\]

where the true data \( \hat{w}^k \) is a trajectory of a true model \( \mathcal{B} \in \mathcal{L}^q_n \) and the measurement noise \( \hat{w}^k \) is zero mean normally distributed measurement noise with variance \( \sigma^2(v)^k \), i.e., up to a scaling factor, the weights \( v \) are the inverse of the noise covariance. Under additional mild assumptions,
the estimator $\hat{B}^*$ is consistent and the parameters have asymptotically normal joint distribution [18], [19].

B. Functions of the package

The function ident solves the approximate identification problem (SYSID), and misfit computes the misfit $M(\bar{w}_d, \bar{B})$. They implement the following mappings:

ident: $(\bar{w}_d, m, \ell) \rightarrow \bar{B}$, where $\bar{B}$ (specified by either input/state/output or kernel representation) is a locally optimal solution of (SYSID)

misfit: $(\bar{w}_d, \bar{B}) \rightarrow (M, \hat{\bar{w}})$, where $M$ is the misfit between $\bar{B}$ (specified by either input/state/output or kernel representation) and $\bar{w}_d$, and $\hat{\bar{w}}$ is the optimal approximation of $\bar{w}_d$ within $\bar{B}$ (the smoothed trajectory)

$[M, \hat{\bar{w}}, \text{xini}] = \text{ident}(\bar{w}, m, \ell, \text{opt})$

Main parameters:

- $\bar{w}$ is the given set of time series $\bar{w}_d$ — a real MATLAB array of dimension $T \times q \times N$, where $T$ is the number of samples, $q$ is the number of variables, and $N$ is the number of time series. In case of multiple experiments of different duration, $\bar{w}$ should be specified as a cell array with $N$ cells, each one of which is a $T_i \times q$ matrix containing the $i$th time series, i.e.,

  $w(t, :, k) = (w^k(t))^T$ or $w(k)(t, :) = (w^k(t))^T$.

- $(m, \ell)$ is the complexity specification (input dimension $m$ and lag $\ell$).

- $\text{sys}$ is an input/state/output representation of the identified or validated system $\bar{B}$, given by an ss object, or a parameter $\bar{R}$ of a kernel representation(KER) if opt.ss is set to zero.

- info is a structure, containing output information from the structured low-rank approximation solver: info.$M$ is the misfit $M(\bar{w}_d, \bar{B})$, info.time is the execution time, and info.iter is the number of iterations.

- $M$ is the misfit $M(\bar{w}_d, \bar{B})$.

- wh is the optimal approximating set of time series $\hat{\bar{w}}$.

- xini is the initial condition for wh.

C. Optional parameters

opt is an optional argument specifying exact variables, exact initial conditions, and options for the optimization solver, used by the ident function. The options are passed to the functions ident and misfit as fields of a structure.

- ‘exct’ (default value [ ]) — $q$-dimensional vector or $N$-dimensional cell array with $q$-dimensional vector elements, specifying the indices of the exact variables.

- ‘wini’ — specifies exact initial conditions. If wini $= 0$, exact zero initial conditions are specified, i.e., $\text{col}(0,\hat{\bar{w}}^k) \in \bar{B}$. More generally, wini $= w_{\text{ini}}$ is an $\ell$ samples long trajectory (specified by an $\ell \times q \times N$ array or a $N$-dimensional cell array of $\ell \times q$ matrices), defining initial conditions for the time series $\hat{\bar{w}}$, i.e., $\text{col}(w_{\text{ini}}^k, \hat{\bar{w}}^k) \in \bar{B}$.

- ‘sys0’ — initial approximation: an input/state/output representation of a system, given as an ss object, with $m$ inputs, $p := q - m$ outputs, and order $n := \ell p$; or specified by a minimal kernel parameter, i.e., a $p \times (\ell + 1)q$ matrix Default value is computed by the slra function, using unstructured low-rank approximation.

- Arguments allowing the user to specify different optimization algorithms (‘solver’ and ‘method’), control the displayed information (‘disp’), and change the convergence criteria, see Section III.

- ‘ss’ set to zero disables the conversion of $\bar{R}$ to the state-space representation ($A, B, C, D$).

- ‘n’ — order of the identified model. For multiple output system ($p > 1$), the order of the identified model $\bar{B} \in \mathbb{L}_{n, \ell}$ is a multiple of the number of outputs $n = p\ell$.

A model of order $p(\ell - 1) < n < p\ell$ can be specified by the optional parameter opt.n. In this case, the IDENT package does balanced model order reduction of the identified model $\bar{B}$ to obtain a model of order $n$.

V. APPROXIMATE GREATEST COMMON DIVISOR COMPUTATIONS

The problem of finding the AGCD of several polynomials appears, for example, in blind deblurring and distance to controllability problems. As shown in [13], the AGCD problem can be reformulated as a mosaic-Hankel structured low-rank approximation.

Let $P_n$ denote the space of (real or complex) polynomials of degree $n$. The AGCD problem solved by the package is the following. Given $N$ polynomials $p_1 \in P_{n_1}, \ldots, p_N \in P_{n_N}$ of and degree $d$, find the approximating polynomials, which have a common divisor of degree at least $d$:

minimize over $\hat{p}_1 \in P_{n_1}, \ldots, \hat{p}_N \in P_{n_N}$ $\sum_{k=1}^{N} \|p_k - \hat{p}_k\|_{w_k}^2$

subject to $\deg \text{gcd}(\hat{p}_1, \ldots, \hat{p}_N) \geq d$,

where $\| \cdot \|_{w}$ is a weighted norm on the coefficients of the polynomial. Note that $\deg \text{gcd}(\hat{p}_1, \ldots, \hat{p}_N) \geq d$ if there exists a polynomial of $\hat{h}$ degree $d$ (common divisor) such that

$\hat{p}_1 = \hat{g}_1 \hat{h}, \ldots, \hat{p}_N = \hat{g}_N \hat{h}$. 

The solvers are contained in a sub-directory agcd of the SLRA package. The common interface for calling these functions is

$[\text{ph}, \text{info}] = \text{gcd}_\text{xxx}(p, v, d, \text{opt})$

where the input arguments are:

- $p$ is a cell array containing vectors of coefficients of polynomials $p_k$; the coefficients are given from lowest to highest degree of the monomials;

- $v$ is a cell array of the weight vectors of corresponding length;

- $d$ is the degree of the GCD;
The implemented optimization methods are:

- `gcd_nls`: uses variable projection representation with respect to $\hat{h}$, handles real polynomials;
- `gcd_nls_complex`: uses variable projection representation with respect to $\hat{h}$, handles complex polynomials;
- `gcd_cofe`: uses variable projection representation with respect to $\hat{g}_k$, handles real polynomials;
- `gcd_syl`: uses Sylvester subresultant low-rank approximation, handles real polynomials.

The default initial approximation is computed from a kernel of a Sylvester subresultant matrix. There is an alternative way to compute the initial approximation using the matrix pencil, which is invoked by the function `hini.mp`. For more details on these methods, see [13].

VI. ACKNOWLEDGMENTS

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