An application of system identification in metrology

Ivan Markovsky
Department ELEC
Vrije Universiteit Brussel (VUB)
Pleinlaan 2, Building K
B-1050 Brussels, Belgium
Tel.: +32-2-629.29.42, Fax : +32-2-629.28.50
ivan.markovsky@vub.ac.be

Abstract

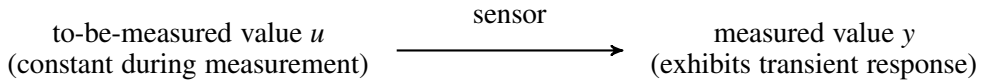
Metrology is advancing by development of new measurement techniques and corresponding hardware. A given measurement technique, however, has fundamental speed and precision limitations. In order to overcome the hardware limitations, we develop signal processing methods based on the prior knowledge that the measurement process dynamics is linear time-invariant.

Our approach is to model the measurement process as a step response of a dynamical system, where the input step level is the quantity of interest. The solution proposed is an algorithm that does real-time processing of the sensor's measurements. It is shown that when the measurement process dynamics is known, the input estimation problem is equivalent to state estimation. Otherwise, the input estimation problem can be solved as a system identification problem. The main underlying assumption is that the measured quantity is constant and the measurement process is a low-order linear time-invariant system. The methods are validated and compared on applications of temperature and weight measurement.

Keywords: system identification, behavioral approach, Kalman filtering, metrology, reproducible research.

1 Introduction

The topic of the paper is a prototypical problem in metrology: a quantity of interest is measured by a measurement device, called a *sensor*. The sensor is a dynamical system with input—the to-be-measured value u (assumed constant during the measurement)—and output—the sensor's reading y (which changes as a function of time):



Two familiar examples are temperature and weight measurement. They are used to motivate the abstract problem formulation and to test the performance of the methods proposed in the paper.

Example 1 (Temperature measurement). A thermometer is moved to a place with temperature $\bar{u}$. The measured temperature y satisfies the Newton's law of cooling

$$\frac{\mathrm{d}}{\mathrm{d}t}y = a\big(y - \bar{u}\big), \tag{1}$$

where a is a negative constant that depends on the thermometer and the environment. The differential equation (1) defines a first order linear time-invariant system with input the environmental temperature $\bar{u}$ and output the thermometer's reading y. Moving the thermometer from one place to another has the effect of a step input, with initial condition the temperature at the first place. The goal is to measure the temperature at the second place while there is still heat exchange between the thermometer and the environment. □

Example 2 (Weight measurement). An object with mass M is placed on a scale with mass m. At the time of placing the object, the scale is in a specified (in general nonzero) initial condition. The object placement has the effect of a step input as well as a step change of the total mass of the system—scale and object. The goal is to measure the object's mass while the system is still in vibration.

In the weight measurement application the sensor is the scale. It is modeled as a mass, spring, damper system

$$(M+m)\frac{\mathrm{d}^2}{\mathrm{d}t^2}y = -ky - d\frac{\mathrm{d}}{\mathrm{d}t}y - Mg, \tag{2}$$

where y is the scale's reading, g is the gravitational constant, k is the elasticity constant, and d is the damping constant. In this case, the sensor is a second order linear time-invariant dynamical system with input the measured mass M. Note that the system's dynamics depends on the measured mass M, which is unknown at the start of the measurement. □

The sensor's measurement y exhibits longer or shorter transient, depending on the sensor and the environment. Physically, the transient represents the exchange of mass or energy, which takes place during the measurement process. In metrology, of interest is the steady-state value $\bar{y} = \lim_{t\to\infty} y(t)$, reached in theory only asymptotically. In practice, the measurement is taken after "sufficient decay" of the transient, *i.e.*, when the error $e_y(t) := \|\bar{y} - y(t)\|$ becomes "small". Therefore, there is a *trade-off* between fast measurement (small measurement time t) and accurate measurement (small error e_y). This trade-off is a *fundamental limitation* of the measurement device determined by the physical laws on the basis of which the sensor is build.

Our goal is to achieve fast and accurate measurement by *predicting* the steady-state value $\bar{y}$ from data $y(t)$ collected over an interval $[0,T]$. The problem is referred to as *dynamic measurement* and the solution is based on digital signal processing. The prediction algorithm, called *compensator*, is a dynamical system with input—the sensor output y—and output—the compensated sensor's reading y_c:

$$\begin{array}{c}\text{sensor output signal } y\\ \text{("long" transient response)}\end{array} \xrightarrow{\text{compensator}} \begin{array}{c}\text{compensated signal } y_\mathrm{c}\\ \text{("short" transient response)}\end{array}$$

Ideally, the compensator completely eliminates the transient. In practice, y_c still exhibits a transient. In addition, *disturbances and measurement noise* affect the sensor measurement y even in a steady-state. The scientific challenge

of dynamic measurement is to design a compensator that achieves simultaneously short transient response and good disturbance and noise filtering.

State-of-the-art

The *classical approach* in dynamic measurement assumes that the process dynamics is *known* and linear time-invariant, see (Eichstädt et al. 2010) for an overview. Consequently, the compensator is also a linear time-invariant system, designed by frequency or time domain de-convolution techniques. The assumption that the process dynamics is known, however, is often unrealistic. In the temperature measurement example, the process dynamics depends on the heat transfer coefficient, which may vary due to unpredictable factors. In the weight measurement example, the process dynamics depends on the unknown mass M. In general, the measurement process dynamics depends on the sensor, which is known, and on the environment or the measured quantity, which are unpredictable or unknown.

In order to deal with the issue of the unknown process dynamics in (Shu 1993; Niedźwiecki and Wasilewski 1996) an adaptive compensator is proposed. Adaptive methods perform simultaneously online model identification and filtering. The methods of (Shu 1993; Niedźwiecki and Wasilewski 1996) are specifically designed for weight measurement. They need nontrivial modifications for other applications. A model-free method, based on ordinary recursive least-squares estimation, is presented in (Markovsky 2015). The computational cost of the model-free method is comparable to the one of a linear time-invariant compensator.

Novelty and contributions

We formulate mathematically the dynamic measurement problem as an input step level estimation problem. Higher order, multivariable measurement processes are considered, which is a generalization over the previously considered in metrology scalar measurement processes. In particular, *sensor fusion* falls into our setting.

In the theoretical development of the methods we use the behavioral approach, where systems are viewed as sets of trajectories rather than equations such as the ones in a transfer function or a state space representation. The ability to switch from one system representation to another is effectively used in the paper to obtain alternative solution methods and to reduce the new problem to solved problems.

In the case of known measurement process (Section 3.1), the input estimation problem is equivalent to a state estimation problem for an augmented autonomous system. The implication of this result is that the Kalman filter, designed for the augmented system is the optimal estimator in the case of Gaussian noise. Deriving the optimal (maximum likelihood) estimator for the dynamic measurement problem with known dynamics is our *first contribution*.

In the case of unknown measurement process (Sections 3.2 and 3.3), the problem is reduced to identification from step response under nonzero initial conditions and identification of an autonomous system with a pole at one. These problems can be solved by existing methods, *e.g.*, the prediction error (Söderström and Stoica 1989; Ljung 1999) and the low-rank approximation methods (Markovsky 2008; Markovsky and Usevich 2014). Deriving the the maximum likelihood estimator for the dynamic measurement problem with unknown dynamics is our *second contribution*.

2 Notation and problem statement

2.1 Notation

A dynamical system is defined by the set of its trajectories. The statement "w is a trajectory of the system $\mathscr{B}$" is concisely written as "$w \in \mathscr{B}$". We assume that the system under consideration has an input-output partitioning $w = (u, y)$, *i.e.*, the first components of the trajectory are inputs and the remaining ones are outputs.

Let σ be the shift operator $(\sigma x)(t) = x(t+1)$. A linear time-invariant system $\mathscr{B}$ admits a state space representation

$$\mathscr{B} = \mathscr{B}_{\mathrm{ss}}(A,B,C,D) := \{\, w = (u,y) \mid \text{there is } x, \text{ such that } \sigma x = Ax + Bu,\ y = Cx + Du \,\}. \tag{3}$$

A state space representation $\mathscr{B}_{\mathrm{ss}}(A,B,C,D)$ is called minimal if it has the smallest possible dimension of the A matrix. This dimension is called the order of the system. The model class of linear time-invariant systems with $q = \mathtt{m} + \mathtt{p}$ external variables, $\mathtt{m}$ of which are inputs, and with order at most n is denoted by $\mathscr{L}^q_{\mathtt{m},n}$. A state space representation of an autonomous linear time-invariant system $\mathscr{B}$ is denoted by $\mathscr{B}_{\mathrm{ss}}(A,C)$.

The notation used in the paper is summarized in Table 1.

symbol	definition
$\mathscr{B}$	model
$w \in \mathscr{B}$	w is a trajectory of $\mathscr{B}$
$w\vert_{[0,T]}$ / $\mathscr{B}\vert_{[0,T]}$	restriction of w / $\mathscr{B}$ on the interval $[0,T]$
$w = (u,y)$	input/output partitioning of w
q / $\mathtt{m}$ / $\mathtt{p}$	number of variables / inputs / outputs
σ	shift operator $(\sigma w)(t) := w(t+1)$
$\mathscr{B}_{\mathrm{ss}}(A,B,C,D)$	input-state-output representation (3)
$\mathscr{B}_{\mathrm{ss}}(A,C)$	autonomous system state space representation
$\mathscr{L}^q_{\mathtt{m},n}$	linear time-invariant systems with at most $\mathtt{m}$ inputs and order at most n
$x \sim \mathrm{N}(m_x, V_x)$	x is a normally distributed random vector with mean m_x and variance V_x
$\mathrm{dcgain}(\mathscr{B})$	dc-gain of $\mathscr{B}$
s	unit step signal
I_n	the $n \times n$ identity matrix
$\mathbf{1}_n$	the $n \times 1$ vector of ones, $\mathbf{1}_n = [\underbrace{1 \ \cdots \ 1}_{n}]^\top$

Table 1: Notation used in the paper.

2.2 The step input estimation problem

The dynamic measurement problem is formalized as an input step level estimation problem for a stable linear time-invariant system $\mathscr{B}$. The observed data is an output trajectory, generated by a step input $u = \bar{u}s$, where $\bar{u}$ is an unknown constant and s is the unit step function:

$$y = y_0 + \widetilde{y}, \qquad \text{where } (\bar{u}s, y_0) \in \mathscr{B} \text{ and } \widetilde{y} \sim \mathrm{N}(0, \xi^2 I). \tag{4}$$

Here y_0 is the "true data" and $\widetilde{y}$ is the measurement noise, which is modeled as a zero mean, white Gaussian stochastic process.

We assume that that the dc-gain $G := \mathrm{dcgain}(\mathscr{B})$ of the system is known and has full column rank. This assumption corresponds in metrological applications to *calibration* of the measurement device. (Note that in Example 1, the dc-gain of the system is equal to one and, in Example 2, the dc-gain $-g/k$ is independent of the measured quantity M.) Since the process is stable, it has a steady-state regime, where the output is a constant $\bar{y} = G\bar{u}$. The full column rank assumption ensures that the value of interest $\bar{u}$ can be determined uniquely from $\bar{y}$.

Note 3 (Calibration and offset). Without calibration, *i.e.*, without prior knowledge of G and without G being full column rank, $\bar{u}$ can not be inferred uniquely. Therefore, we assume that the measurement process dynamics $\mathscr{B}$ is a stable linear time-invariant system with full column rank dc-gain, which is known. We assume also that there is no systematic errors (offset) in the measurements, which also can only be corrected by a proper calibration procedure prior to doing the measurement experiment.

3 Solution methods

3.1 Known model of the measurement process dynamics

By definition, $(\bar{u}s, y)$ is a trajectory of the system $\mathscr{B}_{\mathrm{ss}}(A,B,C,D)$ if there is $x(0) = x_{\mathrm{ini}}$, such that

$$\sigma x = Ax + B\bar{u}s \quad \text{and} \quad y = Cx + D\bar{u}s.$$

Since, for $t \geq 0$, $u(t)$ is a constant $\bar{u}$, we have that $\sigma u = u$ in the discrete-time, with $u(0) = u_{\mathrm{ini}}$. Then, defining the "augmented" state vector $x_{\mathrm{aug}} = \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, we obtain the autonomous model

$$\mathscr{B}_{\mathrm{aut}} := \mathscr{B}_{\mathrm{ss}}(A_{\mathrm{aut}}, B_{\mathrm{aut}}), \qquad \text{where} \quad A_{\mathrm{aut}} := \begin{bmatrix} A & B \\ 0 & I_{\mathtt{m}} \end{bmatrix}, \qquad C_{\mathrm{aut}} := \begin{bmatrix} C & D \end{bmatrix}, \qquad x_{\mathrm{aug}}(0) = \begin{bmatrix} x(0) \\ \bar{u} \end{bmatrix}. \tag{5}$$

The step response data $(\bar{u}s, y)$ is a trajectory of the system $\mathscr{B}_{\mathrm{ss}}(A,B,C,D)$ if and only if the output y is a trajectory of the autonomous system $\mathscr{B}_{\mathrm{ss}}(A_{\mathrm{aut}}, B_{\mathrm{aut}})$. Therefore, the input estimation problem with known model can be solved as a state estimation problem for the augmented autonomous system $\mathscr{B}_{\mathrm{aut}}$. In discrete-time, with exact data (no measurement noise), a dead-beat observer (Bisiacco, Valcher, and Willems 2006) computes the exact state vector in at most $n_{\mathrm{f}} := n + \mathtt{m}$ samples. In the case of noisy data, the maximum likelihood state estimator is given by the Kalman

filter for the augmented model $\mathscr{B}_{\mathrm{ss}}(A_{\mathrm{aut}}, B_{\mathrm{aut}})$. The computational cost is $O(n_{\mathrm{f}}^2 + n_{\mathrm{f}}\mathtt{p})$ floating point operations per step, provided the filter gain is pre-computed off-line. More efficient algorithms that avoid the increase in the state dimension from n to n_{f} are described in (Willman 1969; Friedland 1969).

Corresponding to the measurement error model (4), we have the state space representation

$$\sigma x_{\mathrm{aug}} = A_{\mathrm{aut}} x_{\mathrm{aug}}, \qquad y = C_{\mathrm{aut}} x_{\mathrm{aug}} + \widetilde{y}.$$

The maximum-likelihood estimator for this model is given by the time-varying Kalman filter (Kailath, Sayed, and Hassibi 2000), summarized in Algorithm 1. The initial state $x_{\mathrm{aug}}(0)$ and covariance $P(0)$ encapsulate available prior

Algorithm 1 Maximum-likelihood dynamic measurement with known model based on Kalman filtering.

Input: Model $\mathscr{B}_{\mathrm{ss}}(A,B,C,D)$ of the measurement process, measurement noise standard deviation ξ, initial state $x_{\mathrm{aug}}(0)$, initial covariance matrix $P(0)$, and output observations $y(1),\ldots,y(T)$.

1: **for** $t = 0$ to T do **do**

2: $\quad K(t) = A_{\mathrm{aut}} P(t) C_{\mathrm{aut}}^{\top} (\xi^2 I_{\mathtt{p}} + C_{\mathrm{aut}} P(t) C_{\mathrm{aut}}^{\top})^{-1}$

3: $\quad \widehat{x}_{\mathrm{aug}}(t+1) = A_{\mathrm{aut}} \widehat{x}_{\mathrm{aug}}(t) + K(t)\big(y(t) - C_{\mathrm{aut}} \widehat{x}_{\mathrm{aug}}(t)\big)$

4: $\quad P(t+1) = A_{\mathrm{aut}} P(t) A_{\mathrm{aut}}^{\top} - K(t)\big(\xi^2 I_{\mathtt{p}} + C_{\mathrm{aut}} P(t) C_{\mathrm{aut}}^{\top}\big) K^{\top}(t)$

5: **end for**

Output: $\widehat{u} = \widehat{x}_{\mathrm{aug},n+1}$

knowledge about the measured value $\bar{u}$ and the initial condition $x(1)$. In the simulation examples, we set $x(0) = 0$ and $P(0) = 10^8 I_{n+\mathtt{m}}$.

3.2 Reduction to identification from step response data

With unknown model, the input step value estimation problem resembles the identification problem of finding a linear time-invariant system from step response data. There are however three differences. First, the input is unknown. Second, the dc-gain of the system is constrained to be equal to a given matrix G. Third, the goal of the problem is to find the input rather than the unknown system dynamics. As shown next the first two peculiarities of the input estimation problem are easily dealt with by converting the input estimation problem to an equivalently system identification problem from step response data.

Let $P \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ be a nonsingular matrix, defining the one-to-one relation $u' \leftrightarrow u$, via $u' = P^{-1}u$. Using this relation, we define the system $\mathscr{B}'$ by $\mathscr{B}' := \{\, (u', y) \mid (u, y) \in \mathscr{B} \,\}$. The dc-gain of $\mathscr{B}'$ is $G' = GP$. Indeed,

$$\bar{y} := \lim_{t\to\infty} y(t) = G\bar{u} = \underbrace{GP}_{G'}\underbrace{P^{-1}\bar{u}}_{\bar{u}'} = G'\bar{u}'.$$

The implication of the transformation $u' \leftrightarrow u$ is that instead of considering the system $\mathscr{B}$ with know dc-gain G and unknown input step level $\bar{u}$, we consider the system $\mathscr{B}'$ with a known input step level $\bar{u}'$ and unknown dc-gain G'. Since $\bar{u}'$ can be chosen as an arbitrary nonzero vector, we set $\bar{u}'$ to be the vector of all ones $\mathbf{1}_{\mathtt{m}}$.

With the transition from $\mathscr{B}$ to $\mathscr{B}'$, the nonstandard problem of identifying $\mathscr{B} \in \mathscr{L}^q_{\mathtt{m},n}$ from data $(\bar{u}s, y)$ with unknown input step value $\bar{u}$ and fixed dc-gain G becomes a standard problem of identification of $\mathscr{B}' \in \mathscr{L}^q_{\mathtt{m},n}$ from data $(\mathbf{1}_{\mathtt{m}}s, y)$. Once $\mathscr{B}'$ is obtained from the identification method, the value of interest $\bar{u}$ is found by

1. computing $G' = \text{dcgain}(\mathscr{B}')$,
2. solving the system of linear equations $G' = GP$ for P, and
3. setting $\bar{u} = P\mathbf{1}_{\mathtt{m}}$.

For exact data, the map $(\mathbf{1}_{\mathtt{m}}s, y) \mapsto \mathscr{B}'$ is the exact identification problem of computing the most powerful unfalsified model $\mathscr{B}_{\text{mpum}}(w)$ of $(\mathbf{1}_{\mathtt{m}}s, y)$ in the model class of linear-time invariant systems, *i.e.*, the least complicated exact linear time-invariant model for the data w, (Willems 1986, Definition 4). The problem of fining $\mathscr{B}_{\text{mpum}}(w)$ is also known as the deterministic identification problem (Gopinath 1969; Budin 1971). Subspace algorithms that realize the map $w \mapsto \mathscr{B}_{\text{mpum}}(w)$, *i.e.*, solving the exact identification problem for linear time-invariant model classes are presented in (Van Overschee and De Moor 1996a, Chapter 2) and (Markovsky et al. 2006, Chapter 7).

For noisy data, generated in the output error setup (4), the maximum-likelihood identification problem is

$$\text{minimize} \quad \text{over } \widehat{\mathscr{B}}' \text{ and } \widehat{y} \quad \|y - \widehat{y}\| \quad \text{subject to} \quad (\mathbf{1}_{\mathtt{m}}s, \widehat{y}) \in \widehat{\mathscr{B}}' \in \mathscr{L}^q_{\mathtt{m},n}. \tag{6}$$

The parameter of interest $\widehat{u}$ is then computed by solving the system of linear equations $G\widehat{u} = \text{dcgain}(\widehat{\mathscr{B}}')\mathbf{1}_{\mathtt{m}}$. Problem (6) can be solved by existing algorithms and software, such as the ones in the System Identification Toolbox of Matlab (Ljung 2014). In the simulation examples, we use the identification package (Markovsky 2013) based on Hankel structured low-rank approximation methods. The maximum likelihood method for dynamic measurement based on input-output identification is summarized in Algorithm 2.

Algorithm 2 Maximum likelihood method for dynamic measurement based on input-output identification.

Input: dc-gain G and order n of the measurement process, output observations $y(1), \ldots, y(T)$.

1: **for** $t = 1$ to T do **do**
2: Solve the identification problem (6) for $y = \big(y(1), \ldots, y(t)\big)$.
3: Let $G' := \text{dcgain}(\mathscr{B}')$
4: Solve the system $G\widehat{u} = G'\mathbf{1}_{\mathtt{m}}$.
5: **end for**

Output: $\widehat{u}$

Note 4 (Real-time implementation). There is a trade-off between fast measurement and accurate estimation. Since the process dynamics is not known in advance, we update the estimate while the data is collected. The identification problem (6) on step 2 of the algorithm is solved in real-time, *i.e.*, a recursive implementation of the method is used. For the optimization based methods such as the prediction error methods and the methods based on the structured low-rank approximation, we use as an initial approximation at time t the model obtained at time $t-1$. This does not

yield a truly recursive method because there is no guarantee that the number of iterations needed for convergence is fixed. In practice however, 2–3 iterations (see the results in Figures 2–4, right) are sufficient for obtaining the specified convergence tolerance.

3.3 Solution by identification of an autonomous system

An alternative way of solving the input estimation problem is to exploit its equivalence to autonomous system identification. Consider again the augmented system (5). An equivalent representation of $\mathscr{B}_{\mathrm{ss}}(A_{\mathrm{aut}}, B_{\mathrm{aut}})$ is

$$\mathscr{B}'_{\mathrm{aut}} := \mathscr{B}_{\mathrm{ss}}\left(\begin{bmatrix} A & B\bar{u} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} C & D\bar{u} \end{bmatrix}\right), \qquad x'_{\mathrm{aug}}(0) = \begin{bmatrix} x(0) \\ 1 \end{bmatrix} \tag{7}$$

The system $\mathscr{B}'_{\mathrm{aut}}$ has order $n+1$, and a fixed pole at 1.

For exact data, the map $y \mapsto \mathscr{B}'_{\mathrm{aut}}$ is equivalent to the realization problem and can be solved by Kung's method (Kung 1978). Kung's method is an effective heuristic for solving also the approximate realization problems in the case of noisy data. In the noisy case, however, the identified system, in general, has no fixed pole at 1. The maximum likelihood identification problem is

$$\text{minimize} \quad \text{over } \widehat{\mathscr{B}}'_{\mathrm{aut}} \text{ and } \widehat{y} \quad \|y-\widehat{y}\| \quad \text{subject to} \quad \widehat{y} \in \widehat{\mathscr{B}}'_{\mathrm{aut}} \in \mathscr{L}^{\mathtt{p}}_{0,n+1} \quad \text{and} \quad \widehat{\mathscr{B}}'_{\mathrm{aut}} \text{ has a pole } \lambda = 1. \tag{8}$$

A method for system identification with fixed poles is presented in (Markovsky 2014, Section 7.2) and is implemented in the SLRA software package (Markovsky and Usevich 2014).

Alternatively, a suboptimal method is to 1) preprocess the data y by finite difference $\Delta y := (1-\sigma^{-1})y$, and 2) use standard methods for autonomous system identification for the preprocessed data Δy. Indeed, if y is a trajectory of $\mathscr{B}'_{\mathrm{aut}}$, then Δy is a trajectory of $\mathscr{B}_{\mathrm{ss}}(A,C)$. The identification problem in this case is

$$\text{minimize} \quad \text{over } \Delta\widehat{\mathscr{B}} \text{ and } \widehat{\Delta y} \quad \|\Delta y - \widehat{\Delta y}\| \quad \text{subject to} \quad \widehat{y} \in \Delta\widehat{\mathscr{B}} \in \mathscr{L}^{\mathtt{p}}_{0,n}. \tag{9}$$

Once $\mathscr{B}'_{\mathrm{aut}}$ is identified from the data, the system $\mathscr{B}_{\mathrm{ss}}(A,C)$ is "extracted" from the augmented model $\mathscr{B}'_{\mathrm{aut}}$ by removing the pole $\lambda = 1$. (This can be come by converting the model to a canonical.) In the case of solving (9), we have that $\Delta\mathscr{B} = \mathscr{B}_{\mathrm{ss}}(A,C)$, so that the parameters A and C are directly available from the identified model.

Finally, the parameter of interest $\bar{u}$ is computed from

$$y = \bar{y} + y_{\mathrm{aut}}, \qquad \text{where} \quad \bar{y} = G\bar{u} \quad \text{and} \quad y_{\mathrm{aut}} \in \mathscr{B}_{\mathrm{ss}}(A,C). \tag{10}$$

Using the fact that the columns of the extended observability matrix form a basis for all T samples long trajectories of the system $\mathscr{B}_{\mathrm{ss}}(A,C)$, we obtain the following linear system of equations for the estimation of $\bar{u}$

$$\begin{bmatrix} G & C \\ G & CA \\ \vdots & \vdots \\ G & CA^{T-1} \end{bmatrix} \begin{bmatrix} \bar{u} \\ x_{\mathrm{ini}} \end{bmatrix} = \begin{bmatrix} y(1) \\ \vdots \\ y(T) \end{bmatrix}. \tag{11}$$

The resulting methods, based on the fixed pole identification problem (8) and the finite difference preprocessing (9) are summarized in Algorithms 3 and 4.

Algorithm 3 Maximum likelihood method for dynamic measurement based on autonomous system identification with a fixed pole.

Input: dc-gain G and order n of the measurement process; output observations $y(1),\dots,y(T)$.

1: **for** $t = 1$ to T do **do**
2: Solve the identification problem (8) for $y = \big(y(1),\dots,y(t)\big)$.
3: $\mathscr{B}'_{\text{aut}} \mapsto \mathscr{B}(A,C)$ — removing the fixed pole $\lambda = 1$.
4: Solve the system (11).
5: **end for**

Output: $\widehat{u}$

Algorithm 4 A method for dynamic measurement based on finite differences and autonomous system identification.

Input: dc-gain G and order n of the measurement process; output observations $y(1),\dots,y(T)$.

1: **for** $t = 1$ to T do **do**
2: Let $\Delta y(t) := (1-\sigma^{-1})y(t)$
3: Solve the identification problem (9) for $\Delta y = \big(\Delta y(1),\dots,\Delta y(t)\big)$.
4: Solve the system (11)
5: **end for**

Output: $\widehat{u}$

Note 5 (Statistical properties). The properties of an maximum likelihood estimator, implemented by the methods of Algorithms 2 and 3, are well studied in the literature, see, *e.g.*, (Pintelon and Schoukens 2012). Step response data of a stable system does not allow consistent estimation of the model parameters because the transient process is exponentially decaying (Markovsky and Pintelon 2015). This, however, does not mean that the parameter of interest $\bar{u}$ can not be estimation consistently. Finding a practical consistent estimation method of $\bar{u}$ is an open problem.

Note 6 (Convergence rate). In the case of exact measurements, $2n+1$ data points are sufficient to identify the system exactly (Willems et al. 2005). In the case of noisy data, the convergence rate of the parameters depends on the algorithm and the measurement process. Theoretical results about convergence rates and finite sample size confidence bounds are not available and seem hard to derive. In Section 4, we present empirical results demonstrating the speed-up of the measurement process achieved by the algorithms.

3.4 Model-free method of (Markovsky 2015)

The main idea of the model-free methods (Favoreel 1999; Woodley 2001; Markovsky and Rapisarda 2008), pictorially shown in Figure 1, is to avoid the system identification step in classical model-based methods.

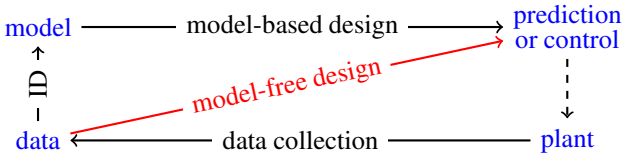


Figure 1: Model-based vs model-free design: the model-based approach requires identification (ID).

The model-free method of (Markovsky 2015) is a subspace-type method (Van Overschee and De Moor 1996b) for solving the autonomous identification problem in Section 3.3, however, it estimates directly the parameter of interest $\bar{u}$. Assuming that Δy is persistently exciting of sufficiently high order (Willems et al. 2005), we have that the most powerful unfalsified model $\mathscr{B}_{\mathrm{mpum}}(\Delta y)$ of Δy is the system $\Delta\mathscr{B}$. Since, by definition,

$$\mathscr{B}_{\mathrm{mpum}}(\Delta y) = \operatorname{span}\big(\{\,\sigma^{\tau}\Delta y \mid \tau \in \mathbb{R}\,\}\big),$$

where "span" is the span (set of all linear combination) of a set of vectors, we have that

$$\mathscr{B}_{\mathrm{mpum}}(\Delta y)|_{[n+1,\ldots,T]} = \operatorname{span}\Bigg(\underbrace{\begin{bmatrix} \Delta y(1) & \Delta y(2) & \cdots & \Delta y(n) \\ \Delta y(2) & \iddots & \iddots & \Delta y(n+1) \\ \vdots & \iddots & & \vdots \\ \Delta y(T-n) & \cdots & & \Delta y(T) \end{bmatrix}}_{H}\Bigg).$$

Then, from (10), we obtain the linear system of equations for $\bar{u}$

$$\begin{bmatrix} \mathbf{1}_{T-n} \otimes G & H \end{bmatrix} \begin{bmatrix} \bar{u} \\ \ell \end{bmatrix} = y|_{[n+1,\ldots,T]}. \tag{12}$$

With exact data, solving equation (12) allows us to compute the value of interest $\bar{u}$ directly from the data y. With noisy data, generically (12) has no solution. A computationally cheap heuristic for estimation of $\bar{u}$ is to solve (12) approximately in the least-squares sense, *i.e.*, obtain the estimate $\widehat{u}$ by solving the optimization problem

$$\text{minimize} \quad \text{over } z \quad \left\| \begin{bmatrix} \mathbf{1}_{T-n} \otimes G & H \end{bmatrix} z - y|_{[n+1,\ldots,T]} \right\|_2, \tag{13}$$

where $\widehat{z} = \left[\begin{smallmatrix} \widehat{u} \\ \ell \end{smallmatrix}\right]$. This results in Algorithm 5.

Algorithm 5 Model-free method for dynamic measurement.

Input: dc-gain G and order n of the measurement process; output observations $y(1), \ldots, y(T)$.

1: Compute $\Delta y := (1 - \sigma^{-1})y$.

2: Solve (13).

Output: $\widehat{u}$.

For real-time data processing, (13) is solved by a recursive least squares method, resulting in computational $O\big((\mathtt{m}+n)^2\mathtt{p}\big)$, which is of the same order of magnitude as the Kalman filtering with precomputed filter gain (see Section 3.1).

4 Experimental validation

For the experimental validation of the methods, we use the temperature and weight measurement applications of Examples 1 and 2, respectively. The data is generated by the output error model (4). $N = 100$ independent noise realizations are generated and the methods listed in Table 2 are applied. Let $\widehat{u}^i(t)$ be the estimate of $\bar{u}$, using the data $\big(y(1),\dots,y(t)\big)$ in an ith Monte Carlo repetition. The average estimation errors

$$e(t) = \frac{1}{N}\sum_{i=1}^{N} \|\bar{u} - \widehat{u}^i(t)\|_1,$$

where $\|\cdot\|_1$ is the 1-norm (sum of absolute values) is used for the validation of the methods.

acronym	method	reference	lines style
NV	"naive" estimator	$u(t) := G^\top (GG^\top)^{-1} y(t)$	black dotted
KF	Kalman filter	Algorithm 1	red dashed-dotted
ML-IO	ML I/O identification	Algorithm 2	blue dashed-circles
ML-FP	ML output-only identification	Algorithm 3	green dashed-squares
ID-FD	identification using Δy	Algorithm 4	green dashed-triangles
DD	model-free method	Algorithm 5	blue dashed

Table 2: Methods compared.

4.1 Temperature measurement

Simulation study

The simulation setup is the temperature measurement application of Example 1 from the introduction. The differential equation (1) defines a first order linear time-invariant dynamical system $\mathscr{B} = \mathscr{B}_{\rm ss}(a,-a,1,0)$ with input $u = \bar{u}s$. The dc-gain of the system is $\text{dcgain}(\mathscr{B}) = 1$ and does not depend on the parameter a. This matches the setup of the input estimation problem in Section 2.2, where the dc-gain is assumed a priori known but the process dynamics is not.

The true output y_0 is obtained with initial condition $x(0) = 1$ and the environmental temperature is $\bar{u} = -1$. We set $a = -0.35$ and collect data over $T = 30$ seconds. The measurement error $\widetilde{y}$ standard deviation is ranging from $\xi = 0.02$ to $\xi = 0.04$. The estimation methods aim to improve the performance of the NV estimator, *i.e.*, the "raw" sensor measurement. Therefore, the average estimation error $e_{\rm nv}$ of the NV method is an upper bonded for the error achieved by the other methods.

In the simulation setup, the KF method is statistically optimal. In addition, it uses prior knowledge about the process dynamics that is not available for the other methods. Its average estimation error $e_{\rm kf}$ gives us a lower bound for the error achieved by the other methods. All methods except for NV use the prior knowledge that the true measurement process is linear time-invariant of order n. The DD and ID-FD methods are statistically suboptimal and are expected to have inferior performance in comparison with the maximum likelihood methods ML-IO and ML-FP.

The above general considerations are confirmed by the results shown in Figures 2–4. The plots in the left column show the average estimation errors e over the interval $[5,30]$, while the plots in the center zoom on the interval $[15,30]$. The average estimation errors of ML-IO and ML-FP are indistinguishable, which confirms that the two methods obtain the same results. Indeed, the ML-IO and ML-FP methods solve equivalent optimization problems, whose solution is the maximum-likelihood estimator.

The performance of the suboptimal methods DD and ID-FD depends on the signal-to-noise (SNR) ratio. For high SNR ($\xi < 0.02$), their performance is close to optimal, however, for decreasing SNR ($\xi > 0.02$), their performance degrades. The DD method is more sensitive to the noise. For noise standard deviation $\xi > 0.04$, its performance is comparable to that of the NV estimator. Note, however, that the DD-method is based on the ordinary recursive least-squares estimation method and is computationally less demanding than the optimization based methods ML-IO, ML-FP, and ID-FD. The average number of iterations of the ML-IO, ML-FP, and ID-FD methods are shown in the right plots of Figures 2–4. On average two iterations are needed for convergence.

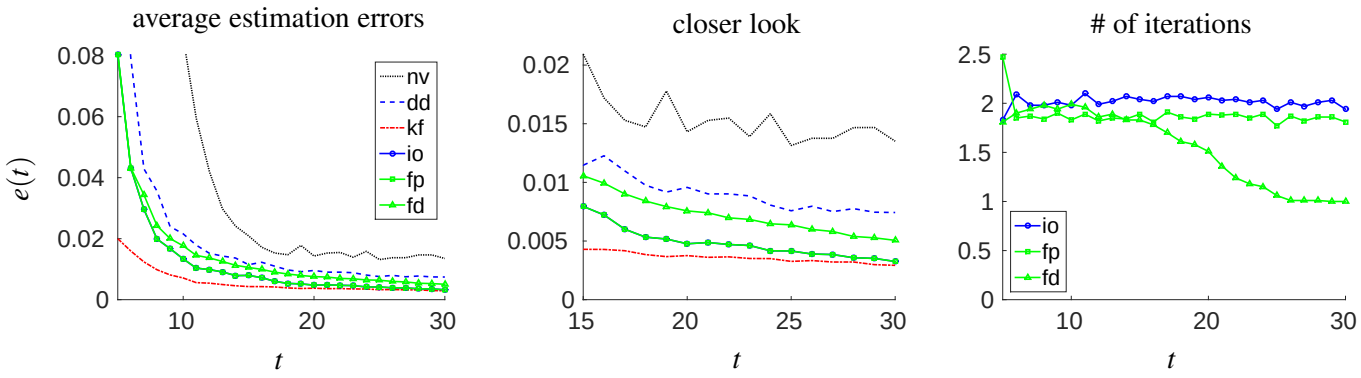


Figure 2: Estimation errors in a temperature measurement example with noise standard deviating $\xi = 0.02$.

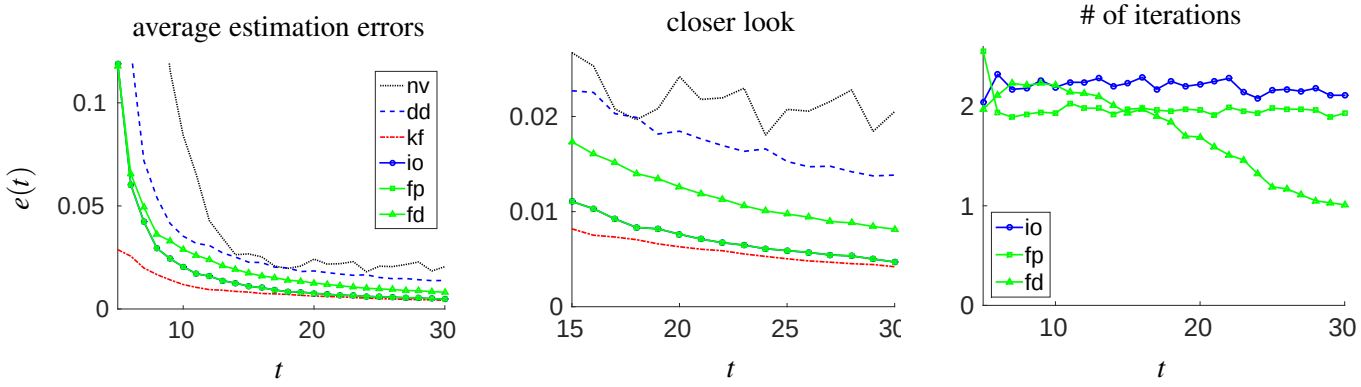


Figure 3: Estimation errors in a temperature measurement example with noise standard deviating $\xi = 0.03$.

Validation on real-life measurements

The methods proposed in the paper are also validated on real data. The experimental setup for the data collection is based on the Lego NXT mindstorms digital signal processor and its digital temperature sensor. The sensor is immersed in water with temperature of 66.7°C. The initial temperature of the thermometer is 35°C. The raw sensor

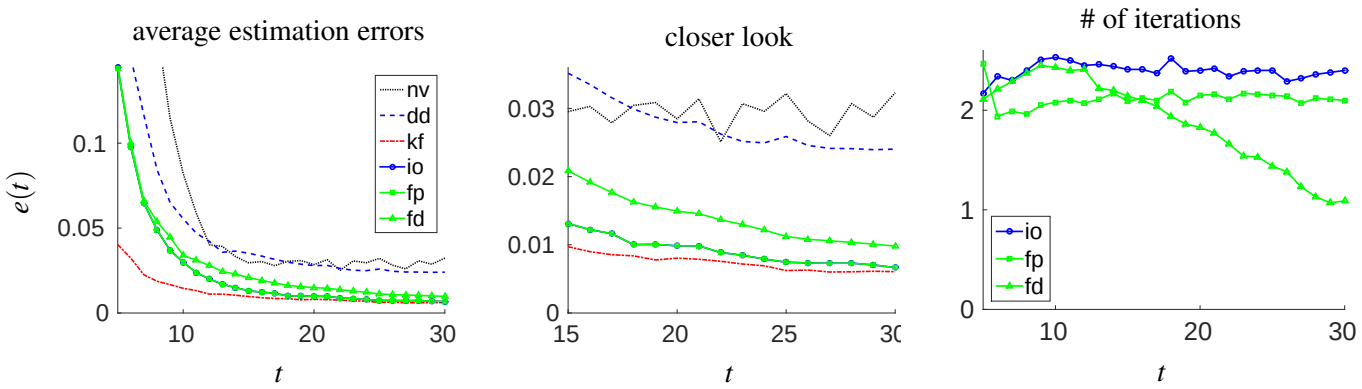


Figure 4: Estimation errors in a temperature measurement example with noise standard deviating $\xi = 0.04$.

measurement y and the optimal fit $\widehat{y}$ by a model $\widehat{\mathscr{B}} \in \mathscr{L}_{1,1}$ obtained from (6) using all the data (80 samples) are shown in figure 5, left.

As a true measured value $\bar{u}$, we take the steady-state temperature $\bar{y}$ obtained from the optimal model $\widehat{\mathscr{B}}$. The KF method is also based on the model $\widehat{\mathscr{B}}$. The model-free algorithm is applied with forgetting factor 0.75. The average estimation errors are shown over the interval $[5,80]$ in Figure 5, center, and over the interval $[15,80]$, in Figure 5, right. The obtained results on real-life measurements show again that the KF method achieves the best performance in terms of both speed and accuracy. The maximum-likelihood methods also show good speed improvement with a estimation error converging to zero. The performance of the ID-FD method however does not match the performance observed in the simulation results. This is due to the combination of quantization errors and use of finite differences. The same problem is observed in the DD method when exponential forgetting is not used. Dealing with the effect of quantization errors in the ID-FD and DD methods is a topic of current research.

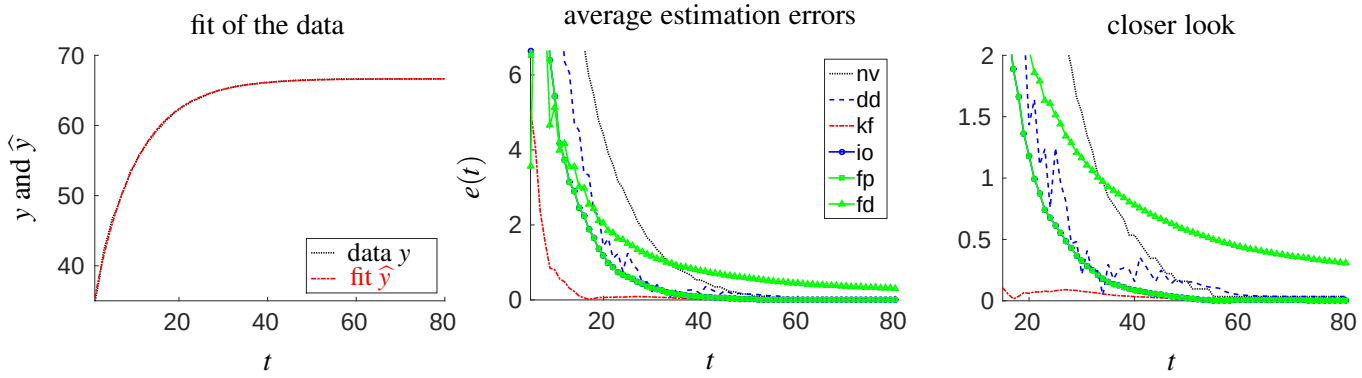


Figure 5: Results on real-life data of temperature measurement.

4.2 Weight measurement

In this section, we use the weight measurement problem described in Example 2 of the introduction. Equation (2) defines a second order linear time-invariant system $\mathscr{B} = \mathscr{B}(A,b,c,0)$, where

$$A = \begin{bmatrix} 0 & 1 \\ k/(M+m) & d/(M+m) \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ -g/(M+m) \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

In the simulation example, the model parameters are chosen as follows:

$$m = 1, \qquad M = 1, \qquad k = 1, \qquad d = 1.1 \quad \text{and} \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Data is collected over $T = 30$ seconds with measurement error $\widetilde{y}$ standard deviation ranging from $\xi = 0.01$ to $\xi = 0.03$.

The results shown in Figures 6–8 are consistent with the ones of the temperature measurement application (*cf.*, Figures 2–4). The best performance (smallest average estimation error e) achieves the Kalman filtering method and the second best performance achieve the ML-IO and ML-FP methods. There is a clear trend for all estimation methods to decrease the average estimation error as the signal-to-noise ratio increases as well as the number of data points increases. In the weight measurement application, the performance difference between the estimation methods and the naive estimator (NV method) is more pronounced.

The average number of iterations needed for the ML-IO, ML-FP, and ID-FD methods is between 3–4 and increases as the signal-to-noise ratio decreases. (Compare with the constant two iterations needed in the temperature measurement application.) In addition, the computational cost per iteration and the minimal number of data points needed for model identification are higher due to the increased model order.

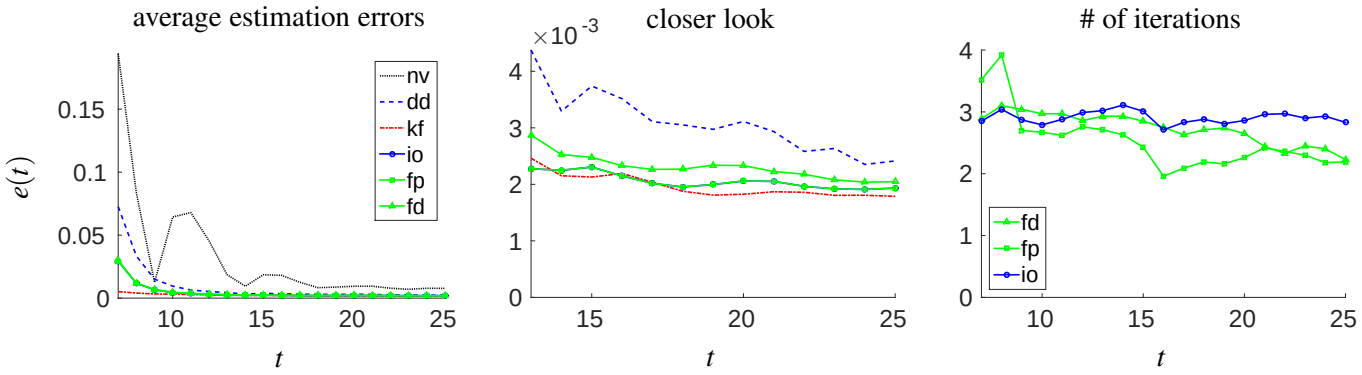


Figure 6: Estimation errors in a mass measurement example with noise standard deviating $\xi = 0.01$.

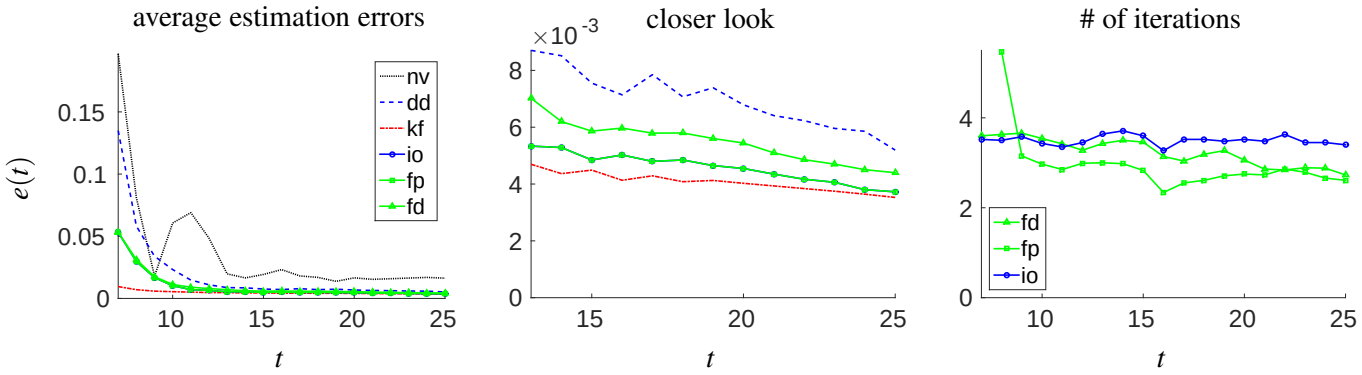


Figure 7: Estimation errors in a mass measurement example with noise standard deviating $\xi = 0.02$.

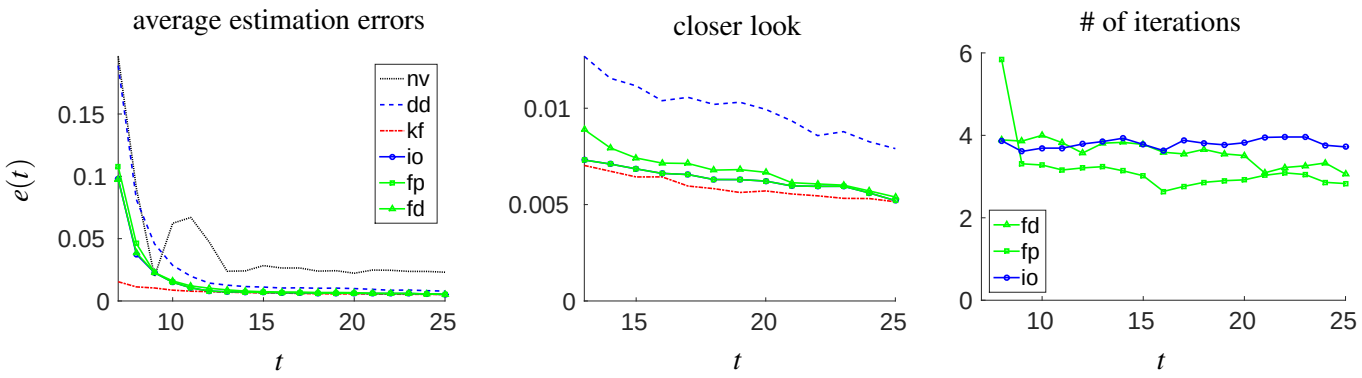


Figure 8: Estimation errors in a mass measurement example with noise standard deviating $\xi = 0.03$.

Implementation and key features of the algorithms

The simulation results reported in the paper are made reproducible in the sense of (Buckheit and Donoho 1995; Hjalmarsson, Rojas, and Rivera 2012). Matlab is used for the implementation of the algorithms and their validation. The necessary files for reproducing the results are available from:

`http://homepages.vub.ac.be/~imarkovs/sensor-cep-experiments.html`

Table 3 shows the Matlab functions implementing the algorithms, the information required, and relative ranking of the computational complexity (lowest is best), and the estimation accuracy (highest is best).

method	implementation	requires	cost	accuracy
"naive" estimator `stepid_nv`	G	1	1	
Kalman filter	`stepid_kf`	$\mathscr{B}$	2	6
ML I/O identification	`stepid_io`	n and G	6	5
ML output-only identification	`stepid_fp`	n and G	5	4
identification using Δy	`stepid_fd`	n and G	4	3
model-free method	`stepid_dd`	n and G	3	2

Table 3: Summary of key features of the algorithms. In the relative ranking of the computational complexity, 1 is the best. In the relative ranking of the estimation accuracy, 6 is the best.

5 Conclusions

Signal processing methods for speed-up of measurement devices are developed. The improvement gained by the methods developed is in both dynamical response as well as in accuracy of the sensor. The requirement for using the methods in practice is a digital signal professing layer attached to the sensor. In the case of an a priori given model, the Kalman filter designed for an augmented autonomous model is a statistically optimal estimation method. It achieves the best performance in both simulation studies and real-life experiments. In the case when a model is not

available, algorithms based on system identification and a model-free method were presented. The main challenges in using system identification for dynamic measurement are: 1) small number of data points, 2) identification from transient response data (step response under nonzero initial conditions), and 3) high computational cost for recursive implementation of the methods on a digital signal processor. These theoretical and algorithmic challenges as well as practical issues such as dealing with quantization errors are a topic of future research.

6 Acknowledgments

Trevor Esward National Physics Laboratory, London) pointed the link of the reported research to the literature on the dynamic measurement problem (Eichstädt et al. 2010). The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement number 258581 "Structured low-rank approximation: Theory, algorithms, and applications".

References

Bisiacco, M., M.-E. Valcher, and J. Willems (2006). "A Behavioral Approach to Estimation and dead-beat observer design with applications to state-space models". In: *IEEE Trans. Automat. Contr.* 51.11, pp. 1787–1797.

Buckheit, J. and D. Donoho (1995). "Wavelets and statistics". In: Berlin, New York: Springer-Verlag. Chap. "Wavelab and reproducible research".

Budin, M. A. (1971). "Minimal realization of discrete linear systems from input-output observations". In: *IEEE Trans. Automat. Control* 16.5, pp. 395–401.

Eichstädt, S. et al. (2010). "Deconvolution filters for the analysis of dynamic measurement processes: a tutorial". In: *Metrologia* 47, pp. 522–533.

Favoreel, W. (1999). "Subspace methods for identification and control of linear and bilinear systems". PhD thesis. ESAT, K.U.Leuven.

Friedland, B. (1969). "Treatment of bias in recursive filtering". In: *IEEE Trans. Automat. Contr.* 14.4, pp. 359–367.

Gopinath, B. (1969). "On the identification of linear time-invariant systems from input-output data". In: *The Bell System Technical J.* 48.5, pp. 1101–1113.

Hjalmarsson, H., Cr. Rojas, and D. Rivera (2012). "Special Section: Wiener-Hammerstein System Identification Benchmark". In: *Control Eng. Practice* 20, pp. 1095–1246.

Kailath, T., A. H. Sayed, and B. Hassibi (2000). *Linear Estimation*. Prentice Hall.

Kung, S. (1978). "A new identification method and model reduction algorithm via singular value decomposition". In: *Proc. 12th Asilomar Conf. Circuits, Systems, Comp.* Pp. 705–714.

Ljung, L. (1999). *System Identification: Theory for the User*. Upper Saddle River, NJ: Prentice-Hall.

— (2014). *System Identification Toolbox: User's guide*. The MathWorks.

Markovsky, I. (2008). "Structured low-rank approximation and its applications". In: *Automatica* 44.4, pp. 891–909.

Markovsky, I. (2013). “A software package for system identification in the behavioral setting”. In: *Control Engineering Practice* 21 (10), pp. 1422–1436. DOI: 10.1016/j.conengprac.2013.06.010.

— (2014). “Recent progress on variable projection methods for structured low-rank approximation”. In: *Signal Processing* 96PB, pp. 406–419. DOI: 10.1016/j.sigpro.2013.09.021.

— (2015). “Comparison of adaptive and model-free methods for dynamic measurement”. In: *IEEE Signal Proc. Letters* 22, pp. 1094–1097. DOI: 10.1109/LSP.2014.2388369.

Markovsky, I. and R. Pintelon (2015). “Identification of linear time-invariant systems from multiple experiments”. In: *IEEE Trans. Signal Process.* 63.13, pp. 3549–3554. DOI: 10.1109/TSP.2015.2428218.

Markovsky, I. and P. Rapisarda (2008). “Data-driven simulation and control”. In: *Int. J. Control* 81.12, pp. 1946–1959. DOI: 10.1080/00207170801942170.

Markovsky, I. and K. Usevich (2014). “Software for weighted structured low-rank approximation”. In: *J. Comput. Appl. Math.* 256, pp. 278–292. DOI: dx.doi.org/10.1016/j.cam.2013.07.048.

Markovsky, I. et al. (2006). *Exact and Approximate Modeling of Linear Systems: A Behavioral Approach*. Monographs on Mathematical Modeling and Computation 11. SIAM. DOI: 10.1137/1.9780898718263.

Niedźwiecki, M. and A. Wasilewski (1996). “Application of adaptive filtering to dynamic weighing of vehicles”. In: *Control Eng. Practice* 4.5, pp. 635–644.

Pintelon, R. and J. Schoukens (2012). *System Identification: A Frequency Domain Approach*. second. Piscataway, NJ: IEEE Press.

Shu, W.-Q. (1993). “Dynamic weighing under nonzero initial conditions”. In: *IEEE Trans. Instrumentation Measurement* 42.4, pp. 806–811.

Söderström, T. and P. Stoica (1989). *System Identification*. Prentice Hall.

Van Overschee, P. and B. De Moor (1996a). *Subspace Identification for Linear Systems: Theory, Implementation, Applications*. Boston: Kluwer.

— (1996b). *Subspace identification for linear systems: Theory, implementation, applications*. Boston: Kluwer.

Willems, J. C. (1986). “From time series to linear system—Part II. Exact modelling”. In: *Automatica* 22.6, pp. 675–694.

Willems, J. C. et al. (2005). “A note on persistency of excitation”. In: *Control Lett.* 54.4, pp. 325–329. DOI: 10.1016/j.sysconle.2004.09.003.

Willman, W. (1969). “On the linear smoothing problem”. In: *IEEE Trans. Automat. Contr.* 14.1, pp. 116–117.

Woodley, B. (2001). “Model free subspace based H_∞ control”. PhD thesis. Stanford University.