# Subspace identification with constraints on the impulse response

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**Abstract**

Subspace identification methods produce unreliable model estimates when a small number of noisy measurements are available. In such cases, the accuracy of the estimated parameters can be improved by using prior knowledge about the system. The prior knowledge considered in this paper is constraints on the impulse response, *e.g.*, steady-state gain, overshoot, and rise time. The method proposed has two steps: 1) estimation of the impulse response with linear equality and inequality constraints, and 2) realization of the estimated impulse response. The problem on step 1 is shown to be a convex quadratic programming problem. In the case of prior knowledge expressed as equality constraints, the problem on step 1 admits a closed form solution. In the general case of equality and inequality constraints, the solution is computed by standard numerical optimization methods. We illustrate the performance of the method on a mass-spring-damper system.

**Keywords:** system identification, subspace methods, prior knowledge, behavioral approach.

## 1 Introduction

Prior knowledge about a system can be expressed naturally as constraints on its behavior, *e.g.*, overshoot and rise time are defined in terms of the step response. In parametric identification, however, the model is represented by a parameter vector—coefficients of a transfer function or a state space representation. The identification problem then becomes a parameter estimation problem and inclusion of the prior knowledge requires its re-formulation as constraints on the parameter vector. This may be nontrivial and leads to more complicated optimization problems. Indeed, linear constraints on the system's behavior often result in nonlinear constraints on the parameter vector.

In this paper, we bypass the difficulties related to inclusion of prior knowledge in parameter estimation by the following a two-step method:

1. estimation of the impulse response, and

2. realization of the estimated impulse response.

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The prior knowledge is imposed on the estimated impulse response in step 1. The method is based on a result from [11], where it is shown that, for exact data, the impulse response of a linear time-invariant system can be computed directly from data by solving an overdetermined system of linear equations. In case of noisy data, generically the system has no solution. Then, a heuristic subspace approach is to estimate the impulse response by solving the system approximately in the least squares sense. We refer to this approach as *data-driven impulse response estimation*.

As shown in the paper, imposing prior knowledge in the method of [11] leads to a convex quadratic programming problem, for which fast and efficient methods exist (the active-set methods [9, Chapter 23], [3, Chapter 5], [5] and the interior point methods [14, 4]). In the case of prior knowledge expressed as equality constraints, the data-driven impulse response estimation problem is a constrained least squares problem and admits a closed form solution (see Section 4.1).

Once the impulse response is estimated, computing the system's parameters of a state-space representation is a classical realization problem [6, 21]. We use Kung's algorithm [8], which involves computation of the singular value decomposition (SVD) and solution of a least squares problem. The overall method requires solution of a least squares problem (in case of equality constraints) or a convex quadratic programming problem (in case of inequality constraints), an SVD, and another least-squares problem. The overall computational cost of the method is comparable to that of classical subspace methods.

In the context of subspace system identification, prior knowledge about stability and passivity of the model is considered in [10, 7]. The approach used in [10, 7] consists in including a regularization term in the least-squares cost function for the estimation of the model parameters. The main result is that, for sufficiently large values of the regularization parameter, the identified model is stable and passive. More recently subspace identification with prior knowledge was considered in [16, 2, 1]. In [2, 1], prior knowledge about the steady-state gain of the system is taken into account by a modified PO-MOESP algorithm, where a constrained least squares problem is solved. The approach of [16] generalizes the regularization approach of [10, 7] to a Bayesian framework for including prior knowledge about the parameter vector. The method however involves a solution of a nonconvex optimization problem, which makes it comparable to the prediction error methods.

The two-step method proposed in this paper is similar to the method of Alenany *et al.* [1]. The latter is also based on estimation of the impulse response by a subspace algorithm, however, it involves a truncation of an infinite sum, which results in inexact impulse response estimate. In contrast, the method proposed here yields exact results when the data is exact under standard identifiability assumptions: persistency of excitation and controllability (Section 3).

Other advantages of the method proposed are the computational cost and simplicity of implementation. The least squares problem in our approach (see (1)) is smaller dimensional than the least squares problem in the method of Alenani *et al.*, see equation (21) in [1]. This is due to the lower block triangular Toeplitz structure of the estimated parameter matrix. In order to take into account the structure Alenani *et al.* vectorize the system of equations. This is at the price of multiplication of the problem dimensions and complicated (Kroneker) structure of the resulting coefficients matrices. In contrast, equation (1) is unstructured least squares problem that can be solved without vectorization. Our method is implement in Matlab and is publicly available. The core part of the code is based on the formula of the analytical solution (3) and is listed in the appendix. The implementation of the method of Alenany for general equality constraint (equation (28) in [1]) seems nontrivial. The authors informed us that it is currently implemented only for prior knowledge in the form of a steady-state gain.

## 2 Preliminaries and notation

The set of real numbers is denoted by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. $\mathbb{R}^{m\times n}$ is the set of $m\times n$ real-valued matrices. "row dim" denotes the number of *block* rows of a matrix or vector, $\operatorname{col}(x_1,\dots,x_n)$ is the column vector with elements $x_1,\dots,x_n$, and $A^+$ denotes the Moore-Penrose pseudoinverse of the matrix $A$. The set of infinite vector-valued time series with $q$ variables is denoted by $(\mathbb{R}^q)^{\mathbb{N}}$.

We use the behavioral notion of a dynamical system: a discrete-time dynamical system $\mathscr{B}$ with $q$ is a subset of the signal space $(\mathbb{R}^q)^{\mathbb{N}}$, see [15]. The notation $\mathscr{B}|_{[t_1,t_2]}$ stands for the restriction of the behavior on the interval $[t_1,t_2]$, *i.e.*,

$$\mathscr{B}|_{[t_1,t_2]} := \{\, w\in(\mathbb{R}^q)^{t_2-t_1} \mid \text{ there are } w_{\text{p}} \text{ and } w_{\text{f}}, \text{ such that } \operatorname{col}(w_{\text{p}},w,w_{\text{f}})\in\mathscr{B}\,\}.$$

We assume that a trajectory $w$ of $\mathscr{B}$ has an input/output partition $w=\left[\begin{smallmatrix}u\\y\end{smallmatrix}\right]$, with $\mathtt{m} := \operatorname{row dim}(u)$ inputs and $\mathtt{p} := \operatorname{row dim}(y)$ outputs, where $q=\mathtt{p}+\mathtt{m}$. In general, a permutation $\Pi w$ of the variables is needed in order to have all inputs as the first variables.

A finite dimensional linear time-invariant system $\mathscr{B}$ is a closed shift-invariant subspace of $(\mathbb{R}^q)^{\mathbb{N}}$. The input/state/output representation of a linear time-invariant system $\mathscr{B}$ is denoted by

$$\mathscr{B}_{\text{i/s/o}}(A,B,C,D) := \{\,(u,y) \mid \text{there is } x \text{ such that } \dot{x}=Ax+Bu \text{ and } y=Cx+Du\,\}.$$

The *order* of the system is the smallest state dimension $\mathtt{n}=\operatorname{row dim}(x)$. The *lag* of a linear time-invariant system is the observability index of the system [13].

The Hankel matrix with $t$ block rows, composed of the sequence $w\in(\mathbb{R}^q)^T$ is denoted by

$$\mathscr{H}_t(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-t+1) \\ w(2) & w(3) & \cdots & w(T-t+2) \\ w(3) & w(4) & \cdots & w(T-t+3) \\ \vdots & \vdots & & \vdots \\ w(t) & w(t+1) & \cdots & w(T) \end{bmatrix}.$$

The time series $u=\big(u(1),\dots,u(T)\big)$ is persistently exciting of order $L$ if the Hankel matrix $\mathscr{H}_L(u)$ is of full row rank.

In general, quantities marked with bar (*e.g.*, $\bar{h}$) refer to noise free or true values. Quantities marked with hat (*e.g.*, $\widehat{h}$) refer to estimates of the true values, and quantities marked with tilde (*e.g.*, $\widetilde{h}$) refer to estimation errors, or perturbation on the true value due to noise.

## 3 Impulse response estimation

In this section, we consider the problem of computing the first $\delta+1$ samples

$$h = \big(h(0), h(1), \dots, h(\delta)\big), \qquad h(t) \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$$

of the impulse response of a linear time-invariant system $\mathscr{B}$ from a finite trajectory

$$w_{\mathrm{d}} := \big(w_{\mathrm{d}}(1), \dots, w_{\mathrm{d}}(T)\big)$$

of the system.

**Lemma 1** ([20])**.** *Let $\mathscr{B}$ be a linear time-invariant system of order $\mathtt{n}$ with lag $\ell$. Under the following assumptions:*

1. *the data $w_d$ is exact,* i.e., *$w_d \in \mathscr{B}$,*
2. *system is $\mathscr{B}$ is controllable, and*
3. *the input component $u_d$ of the trajectory $w_d$ is persistently exciting of order $\mathtt{n}+\ell+1$,*

*the Hankel matrix $\mathscr{H}_t(w_d)$, spans the space $\mathscr{B}|_{[1,t]}$ of all $t$-samples long trajectories of the system $\mathscr{B}$,* i.e.,

$$\operatorname{image}\big(\mathscr{H}_t(w_d)\big) = \mathscr{B}|_{[1,t]}.$$

Lemma 1 implies that there exists a matrix $g$, such that

$$\mathscr{H}_t(y_{\mathrm{d}})g = h.$$

Therefore, the problem of computing the impulse response $h$ from the data $w_{\mathrm{d}}$ reduces to the one of finding the vector $g$ in the above equation.

Define the matrices $U_{\mathrm{p}}$, $U_{\mathrm{f}}$, $Y_{\mathrm{p}}$, $Y_{\mathrm{f}}$ as follows

$$\mathscr{H}_{\ell+\delta+1}(u_{\mathrm{d}}) =: \begin{bmatrix} U_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix}, \qquad \mathscr{H}_{\ell+\delta+1}(y_{\mathrm{d}}) =: \begin{bmatrix} Y_{\mathrm{p}} \\ Y_{\mathrm{f}} \end{bmatrix},$$

where

$$\operatorname{row}\dim(U_{\mathrm{p}}) = \operatorname{row}\dim(Y_{\mathrm{p}}) = \ell$$

and

$$\operatorname{row}\dim(U_{\mathrm{f}}) = \operatorname{row}\dim(Y_{\mathrm{f}}) = \delta+1.$$

Then, under the conditions of Lemma 1, the system of equations

$$\underbrace{\begin{bmatrix} U_{\mathrm{p}} \\ U_{\mathrm{f}} \\ Y_{\mathrm{p}} \end{bmatrix}}_{\mathscr{A}} g = \underbrace{\begin{bmatrix} 0_{\mathtt{m}\ell\times\mathtt{m}} \\ \begin{bmatrix} I_{\mathtt{m}} \\ 0_{\mathtt{m}\delta\times\mathtt{m}} \end{bmatrix} \\ 0_{\mathtt{p}\ell\times\mathtt{m}} \end{bmatrix}}_{v}, \tag{1}$$

is solvable for $g$, and for any particular solution, the matrix $Y_\mathrm{f} g$ contains the first $\delta+1$ samples of the impulse response of $\mathscr{B}$, *i.e.*,

$$Y_\mathrm{f} g = h.$$

This gives Algorithm 1 for the computation of the impulse response from a general trajectory of the system.

**Algorithm 1** Impulse response estimation. `uy2hblk.m`

**Input:** Trajectory $w_\mathrm{d}$, system lag $\ell$, and number of impulse response samples $\delta$.

1: Solve the system of equations (1) and let $g$ be the computed solution.

2: Let $h = Y_\mathrm{f} g$.

**Output:** $h$.

*Note* 2 (Recursive computation of the impulse response). Algorithm 1 finds the first $\delta$ samples of the impulse response; however, the persistency of excitation condition imposes a limitation on how big $\delta$ can be. This limitation can be avoided by a modification of the algorithm that computes iteratively overlapping blocks of $\ell+1$ consecutive samples of $h$ and reconstructs the full $h$ from them.

# 4 Imposing prior knowledge

Next, we consider prior knowledge about the impulse response in the form of equality

$$Eh = f \tag{2}$$

and inequality

$$E'h \leq f'$$

constraints.

## 4.1 Impulse response estimation with equality constraints

From (2), we have that $g \in (EY_\mathrm{f})^+ f + \mathrm{null}(EY_\mathrm{f})$. Therefore, $g$ is of the form

$$g = (EY_\mathrm{f})^+ f + Nz, \quad \text{for some } z,$$

where the columns of $N$ form a basis for the null space of $EY_\mathrm{f}$. From (1), we have

$$\mathscr{A}\big((EY_\mathrm{f})^+ f + Nz\big) = v,$$

where $\mathscr{A}$ and $v$ are defined in (1). Therefore,

$$z = (\mathscr{A}N)^+\big(v - \mathscr{A}(EY_\mathrm{f})^+ f\big).$$

Finally, for the impulse response, we have

$$h = Y_\mathrm{f}\left((EY_\mathrm{f})^+ f + N(\mathscr{A}N)^+\left(v - \mathscr{A}(EY_\mathrm{f})^+ f\right)\right). \tag{3}$$

### 4.2 Impulse response estimation with equality and inequality constraints

In the case of equality and inequality constraints, the problem is a quadratic program

$$\begin{aligned} &\text{minimize} \quad \text{over } g \quad \|\mathscr{A} g - v\| \\ &\text{subject to} \quad E Y_{\mathrm{f}} g = f \quad \text{and} \quad E' Y_{\mathrm{f}} g \leq f'. \end{aligned} \tag{4}$$

It does not admit an analytical solution but due to convexity it can be solved globally and efficiently. We use an active-set algorithm [5], which is implemented in the function `lsqlin` of Matlab. The resulting method for impulse response estimation with linear equality/inequality constraints is summarized in Algorithm 2. Matlab code that implements the algorithm is listed in the appendix.

**Algorithm 2** Impulse response estimation with linear equality/inequality constraints. `uy2h_pk.m`

**Input:** Trajectory $w_{\mathrm{d}}$, system lag $\ell$, equality $(E, f)$ and inequality $(E', f')$ constraints, and number of impulse response samples $\delta$.

**if** there are equality constraints only **then**
    Compute $h$ via (3).
**else**
    Solve the quadratic program (4).
    Let $h = Y_{\mathrm{f}} g$
**end if**

**Output:** $h$.

## 5 Impulse response realization

In this section, we consider the realization problem

$$h \mapsto (\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$$

with

$$h = \big(h(0), h(1), \ldots, h(\delta)\big), \qquad h(t) \in \mathbb{R}^{\mathtt{p} \times \mathtt{m}}.$$

Let $\sigma h$ be the shifted impulse response

$$\sigma h := \big(h(1), h(2), \ldots, h(\delta)\big).$$

The realization algorithm involves a user defined parameter $L$—the number of block rows of the Hankel matrix $\mathscr{H}_L(h)$. With $L$ and $T - L + 1$ greater than or equal to the order $\mathtt{n}$ of the system, we have that rank $\big(\mathscr{H}_L(\sigma h)\big) = \mathtt{n}$. The first step of the algorithm is a rank revealing factorization of the Hankel matrix

$$\mathscr{H}_L(\sigma h) = \mathscr{O}\mathscr{C}.$$

The second step is and solution of the equation

$$\sigma\mathscr{O}\widehat{A} = \sigma^*\mathscr{O}, \tag{5}$$

where $\sigma\mathscr{O}$ removes the first row of $\mathscr{O}$ and $\sigma^*\mathscr{O}$ removes the last row of $\mathscr{O}$. It can be shown that (5) has a unique solution $\widehat{A}$ and the matrices $(\widehat{A},\widehat{B},\widehat{C},\widehat{D})$, where

$$\widehat{D} := h(0), \quad \widehat{C} := \text{the first row of } \mathscr{O}, \quad \text{and} \quad \widehat{B} := \text{the first column of } \mathscr{C},$$

define a realization $\mathscr{B}_{\text{i/s/o}}(\widehat{A},\widehat{B},\widehat{C},\widehat{D})$ of $h$. Therefore, up to a similarity transformation, they are equal to the true system parameters $(\bar{A},\bar{B},\bar{C},\bar{D})$.

With non-exact impulse response $h$, the Hankel matrix $\mathscr{H}_L(\sigma h)$ is generically full rank. An approximate realization algorithm, known as Kung's algorithm, proceeds by computing an approximate rank revealing factorization

$$\mathscr{H}_L(\sigma h) \approx \mathscr{O}\mathscr{C},$$

using the singular value decomposition of $\mathscr{H}_L(\sigma h)$

$$\mathscr{H}_L(\sigma h) = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^\top & V_2^\top \end{bmatrix}, \tag{6}$$

where $U_1 \in \mathbb{R}^{L\times \mathtt{n}}$, $\Sigma_1 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $V_1 \in \mathbb{R}^{L\times\mathtt{n}}$ and defining

$$\mathscr{O} = U_1\sqrt{\Sigma_1} \quad \text{and} \quad \mathscr{C} = \sqrt{\Sigma_1}V_1^\top.$$

Then, the parameters $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, and $\widehat{D}$ are defined via (5) and (5) but contrary to the exact case, equation (5) generically has no solution. An approximate solution is computed via the least squares, total least squares [12], or weighted least squares methods. The weighted least squares method uses as a weight matrix information obtained from the perturbation analysis of the singular value decomposition [17, 18].

*Note* 3 (Choice of the hyper parameter $L$). Empirical results show that square Hankel matrix $\mathscr{H}_L(h)$ leads to best approximation of the given $h$ by the impulse response of the computed realization $\mathscr{B}_{\text{i/s/o}}(\widehat{A},\widehat{B},\widehat{C},\widehat{D})$.

The resulting realization method is summarized in Algorithm 3. The overall algorithm for subspace identification with prior knowledge on the impulse response is a combination of Algorithms 2 and 3 and is listed in Algorithms 4.

*Note* 4. In case of noisy data, the estimated impulse response $\widehat{h}$ on step 1 of Algorithm 4 is generically not exactly realizable by a linear time-invariant system with order less than or equal to $\mathtt{n}$. Then, step 2 involves an approximation. The impulse response of the model $\widehat{\mathscr{B}} := \mathscr{B}_{\text{i/s/o}}(\widehat{A},\widehat{B},\widehat{C},\widehat{D})$ is as a result generically not equal to $\widehat{h}$ and may not satisfy the prior knowledge.

**Algorithm 3** Impulse response realization algorithm. `uy2h.m`

**Input:** Impulse response $h$ and system order $\mathtt{n}$.

1: According to Note 3, we set $L := \lceil Tm/(m+p) \rceil$.

2: Compute the SVD (6) of the Hankel matrix $\mathscr{H}_L$.

3: Solve the shift equation (5).

**Output:** Realization $\mathscr{B}_{\text{i/s/o}}(\widehat{A},\widehat{B},\widehat{C},\widehat{D})$.

**Algorithm 4** Subspace identification with prior knowledge on the impulse response. `uy2ss_pk.m`

**Input:** Trajectory $w_{\text{d}}$, equality $(E,f)$ and inequality $(E',f')$ prior knowledge, and system order $\mathtt{n}$.

1: $(w_{\text{d}},(E,f),(E',f')) \mapsto \widehat{h}$ using Algorithm 2, compute an estimate $\widehat{h}$ of the impulse response.

2: $(\widehat{h},\mathtt{n}) \mapsto (\widehat{A},\widehat{B},\widehat{C},\widehat{D})$ using Algorithm 3, compute the parameters of a state space representation of the system.

**Output:** Identified model $\mathscr{B}_{\text{i/s/o}}(\widehat{A},\widehat{B},\widehat{C},\widehat{D})$.

# 6 Numerical experiments

## 6.1 Simulation setup

In the numerical examples we use a mass-spring-damper system

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}y + d\frac{\mathrm{d}}{\mathrm{d}t}y + ky = u,$$

where the model parameters are chosen as $m = 1$, $d = 1$, and $k = 10$. The data is regularly sampled with sampling period 0.2sec from the continuous-time system. We denote with $\bar{\mathscr{B}}$ by discretized true data generating model. The identification data $w_{\text{d}}$ is obtained in the errors-in-variables setting:

$$w_{\text{d}} = \bar{w} + \widetilde{w}, \quad \text{where } \bar{w} \in \bar{\mathscr{B}}, \text{ and } \widetilde{w} \text{ is zero mean,white Gaussian process with covariance } s^2 I.$$

Here $\bar{w}$ is the “true value” of the trajectory $w_{\text{d}}$. The input $\bar{u}$ is a zero mean, white, random process with uniform distribution in the interval $[0,1]$. The data consists of $T = 50$ samples and the noise standard deviation is $s = 0.01$.

## 6.2 A single equality constraint

In this subsection, we consider prior knowledge is in the form of a single equality constraint (2), with $E := \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$ and $f = E\bar{h}$, with $\bar{h}$ being the impulse response of $\bar{\mathscr{B}}$. Using the data $w_{\text{d}}$ and the prior knowledge specified by the pair $(E,f)$, we estimate the model $\widehat{\mathscr{B}}$ using Algorithm 4 (`uy2ss_pk`). For comparison, we also estimate models using two alternative methods:

- `uy2ss` — the two-stage method, not using the prior knowledge, and
- `n4sid` — the N4SID method [19], implemented in the Identification Toolbox of Matlab, called with the default parameters.

The experiment is repeated $N = 100$ times with different noise realizations (Monte-Carlo simulation). Let $\widehat{\mathscr{B}}^{(k)}$ be the identified model in the $k$th repetition of the experiment and let $\widehat{h}^{(k)}$ be the estimated impulse response. Denote with $\|\mathscr{B}\|$ the 2-norm of the system $\mathscr{B}$. The average relative estimation errors

$$e_{\mathscr{B}} = \frac{1}{N}\sum_{k=1}^{N} \frac{\|\bar{\mathscr{B}} - \widehat{\mathscr{B}}^{(k)}\|}{\|\bar{\mathscr{B}}\|} \qquad \text{and} \qquad e_h = \frac{1}{N}\sum_{k=1}^{N} \frac{\|\bar{h} - \widehat{h}^{(k)}\|}{\|\bar{h}\|}$$

of, respectively, the identified system and the identified first $\delta = 10$ samples of the impulse response are reported in Table 1. In both performance measures—$e_{\mathscr{B}}$ and $e_h$— `uy2ss_pk` improves the results of `uy2ss`, however, using a single equality constraints, in this simulation example `n4sid` produces a better result than `uy2ss_pk`. In the next section, we show that with two or more equality constraints `uy2ss_pk` outperforms `n4sid`.

Table 1: Average relative estimation errors $e_{\mathscr{B}}$ and $e_h$ for the subspace method using one equality constraint as a prior knowledge `uh2ss_pk`, not using the prior knowledge `uh2ss`, and for the N4SID method `n4sid`.

| | uh2ss_pk | uh2ss | n4sid |
|---|---|---|---|
| $e_{\mathscr{B}}$ | 0.1925 | 0.2218 | 0.1597 |
| $e_h$ | 0.1828 | 0.2137 | — |

### 6.3 Multiple equality constraints

The result of the Monte-Carlo simulation in Section 6.2 shows that prior knowledge in the form of a single equality constraint improves the estimation accuracy of both the impulse response as well as the identified system. In this section, we show the estimation errors as a function of the number of equality constraints. We use the simulation setup described in Section 6.1. The matrix $E$ is now chosen as a random $i \times \delta$ matrix, where $i$ is the number of equality constraints and $\delta$ is the number of the estimated impulse response coefficients. The results in Figure 1 shows that the estimation error for the subspace method using the prior knowledge reduces to zero as the number of equality constraints become $i = \delta = 10$. Indeed, in the case $i \geq \delta$, $\bar{h}$ can be computed by solving the system of linear equations $Eh = f$, without using the (noisy) data $w_{\mathrm{d}}$.

### 6.4 Inequality constraints

In this section we consider prior knowledge in the form of upper and lower bounds on the impulse response. The simulation setup is as described in Section 6.1, however, now $\delta = 20$ samples of the impulse response are estimated and the trajectory $w_{\mathrm{d}}$ has $T = 100$ samples. The prior knowledge about the impulse response and the true impulse response $\bar{h}$ are shown in Figure 2, left. On the same plot are superimposed the estimated impulse responses by Algorithms 2 (`uy2hk`) and 3 (`uy2h_pk`). Figure 2, right shows the true impulse response and the impulse responses of the models identified by the three methods compared: `uh2ss_pk`, `uh2ss`, and `n4sid`. The numerical values of

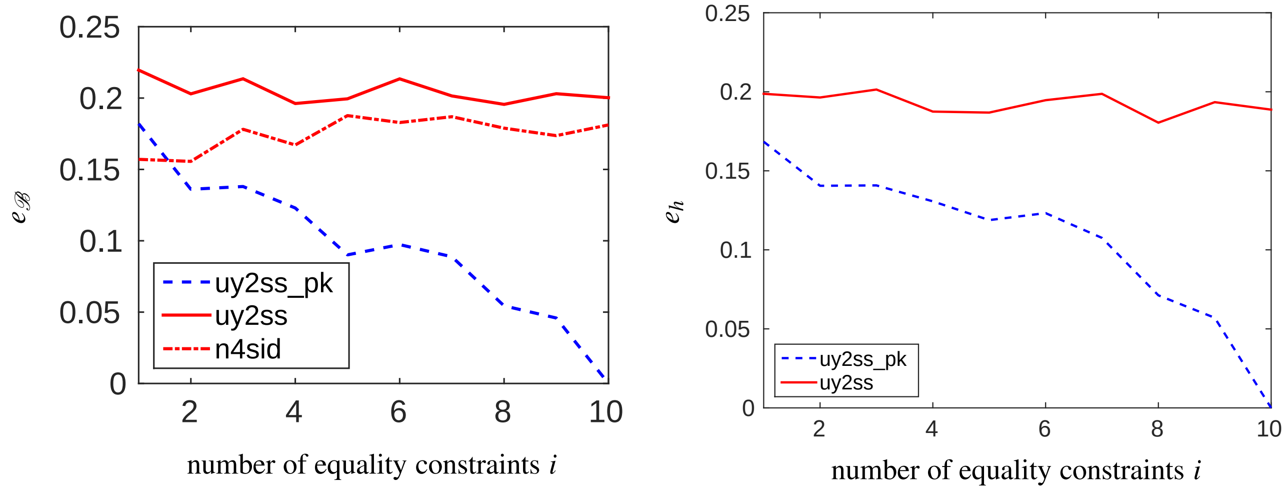


Figure 1: Average relative estimation errors $e_{\mathscr{B}}$ and $e_h$ as a function of the number of equality constraints $i$.

the average relative estimation errors $e_{\mathscr{B}}$ and $e_h$ are given in Table 2. The results empirically confirm the advantage of using the given prior knowledge on the impulse response for the overall identification problem.

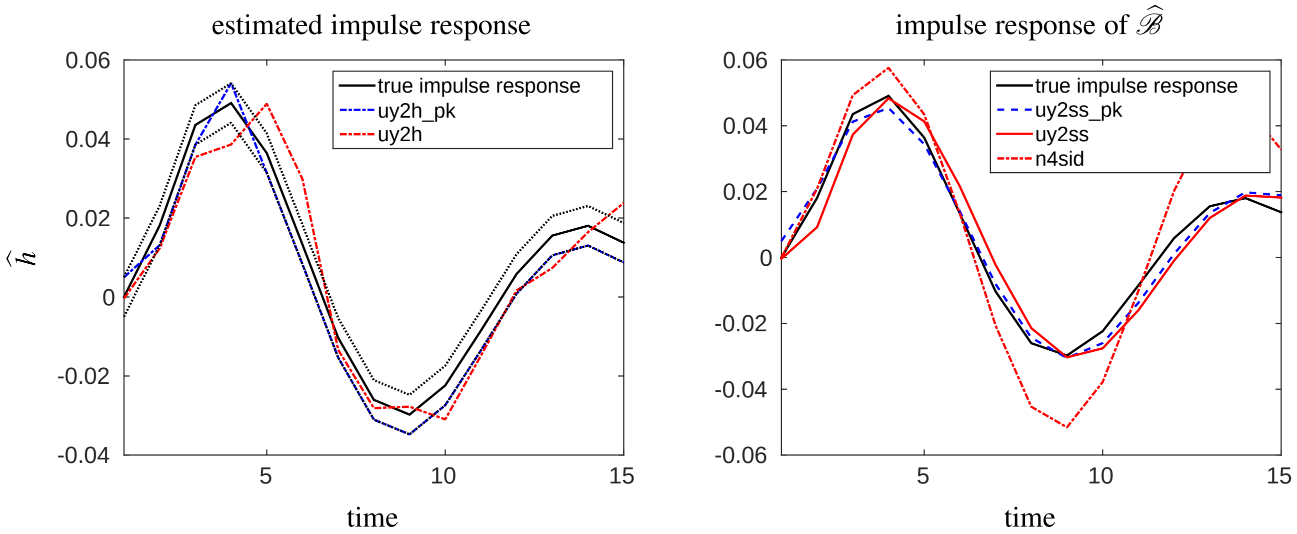


Figure 2: Estimation with inequality constraints. Left — true impulse response (solid line), prior knowledge (dotted lines), and estimates by Algorithm 3 with (`uy2h_pk`) and without (`uy2h`) prior knowledge; Right — estimated impulse responses.

## 7 Conclusions

We considered prior knowledge about the to-be-identified model in the form of linear equality and inequality constraints on the impulse response. The method proposed has two steps: 1) estimation of the impulse response, and 2) realization of the estimated impulse response. Using the data-driven method for impulse response estimation of [11], incorporating prior knowledge involves solution of a convex quadratic programming problem. In case of equal-

Table 2: Average relative estimation errors $e_{\mathscr{B}}$ and $e_h$ for the methods using inequality constraints as a prior knowledge (`uh2ss_pk`), and not using the prior knowledge (`uh2ss` and `n4sid`).

| | uh2ss_pk | uh2ss | n4sid |
|---|---|---|---|
| $e_{\mathscr{B}}$ | 0.2291 | 0.2611 | 1.4984 |
| $e_h$ | 0.2022 | 0.3235 | — |

ity constraints only, the problem allows an analytic solution. In the more general case of equality and inequality constraints, the problem can be solved globally and efficiently by existing optimization methods. The resulting algorithm has computational complexity that is comparable to that of classical subspace algorithms. Numerical examples show the improved estimation accuracy as a result of using the prior knowledge. Statistical analysis of the proposed computational method (consistency, uncertainty bound, *etc*.) is a topic of future work.

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## A Implementation of the impulse response estimation method in Matlab

The method proposed in this paper is implemented and is available from:

`http://homepages.vub.ac.be/~imarkovs/software/detss_pk.tar`

The function implementing the algorithm for impulse response estimation with prior knowledge is `uy2h_pk`. Its input parameters are the observed input `u` and output `y` data of the system, the system's lag `ell`, the number of to-be-estimated impulse response coefficients `delta`, and the equality and inequality constraints, specified by the matrices ( `e`, `f` ) and ( `ep`, `fp` ), respectively. In the actual Matlab code, listed below, the function `blkhank(y,t)` constructs the block-Hankel matrix $\mathscr{H}_t(y)$.

```
function h = uy2h_pk(u, y, ell, delta, e, f, ep, fp)
if (~exist('e') || isempty(e)) && (~exist('ep') || isempty(ep)) % no prior
  h = uy2hblk(u, y, ell, delta); return % Algorithm 1
end
```

```
% definitions
[T, m] = size(u); [T, p] = size(y);
Y  = blkhank(y, ell + delta);
Yp = Y(1:ell * p, :);
Yf = Y(ell * p + 1:end, :);
a = [blkhank(u, ell + delta); Yp];
b = zeros(size(a, 1), m); b(ell * m + 1:(ell + 1) * m, :) = eye(m);

% analytical solution
eYf = e * Yf; Gp = pinv(eYf) * f; N = null(eYf);
g = Gp + N * pinv(a * N) * (b - a * Gp);
% quadratic programming
if (exist('ep') && ~isempty(ep))
  g = lsqlin(a, b, ep * Yf, fp, eYf, f, [], [], g);
end
h = Yf * g;
```

## References


[1] A. Alenany et al. "Improved subspace identification with prior information using constrained least squares". In: *IET Control Theory and Applications* 5 (2011), pp. 1568–1576.

[2] A. Alenany et al. "Subspace identification with prior steady-state information". In: *Proceedings of the International Conference on Computer Engineering and Systems*. Cairo, Egypt, 2010.

[3] Å. Björck. *Numerical Methods for Least Squares Problems*. SIAM, 1996.

[4] R. Byrd, M. Hribar, and J. Nocedal. "An Interior Point Algorithm for Large-Scale Nonlinear Programming". In: *SIAM J. Optim.* 9.4 (1999), pp. 877–900.

[5] P. E. Gill, M. Murray, and M. H. Wright. *Practical Optimization*. Academic Press, 1999.

[6] B. L. Ho and R. E. Kalman. "Effective construction of linear state-variable models from input/output functions". In: *Regelungstechnik* 14.12 (1966), pp. 545–592.

[7] Goethals I. et al. "Identification of Positive Real Models in Subspace Identification by using Regularization". In: *IEEE Trans. Automat. Contr.* 48 (2003), pp. 1843–1847.

[8] S. Kung. "A new identification method and model reduction algorithm via singular value decomposition". In: *Proc. 12th Asilomar Conf. Circuits, Systems, Comp.* 1978, pp. 705–714.

[9] C. Lawson and R. Hanson. *Solving Least Squares Problems*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 1987.

[10] J. Maciejowski. “Guaranteed stability with subspace methods”. In: *Control Lett.* 26 (1995), pp. 153–156.

[11] I. Markovsky and P. Rapisarda. “Data-driven simulation and control”. In: *Int. J. Control* 81.12 (2008), pp. 1946–1959.

[12] I. Markovsky and S. Van Huffel. “Overview of total least squares methods”. In: *Signal Proc.* 87 (2007), pp. 2283–2302. DOI: 10.1016/j.sigpro.2007.04.004. (Citations: 221 in WoS, 410 in GS).

[13] I. Markovsky et al. *Exact and Approximate Modeling of Linear Systems: A Behavioral Approach*. SIAM, 2006. DOI: 10.1137/1.9780898718263. (Citations: 77 in GS).

[14] J. Nocedal and S. Wright. *Numerical optimization*. Springer-Verlag, 1999.

[15] J. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory*. New York: Springer-Verlag, 1998.

[16] P. Trnka and V. Havlena. “Subspace like identification incorporating prior information”. In: *Automatica* 45 (2009), pp. 1086–1091.

[17] R. J. Vaccaro. “A Second-Order Perturbation Expansion for the SVD”. In: *SIAM J. Matrix Anal. Appl.* 15 (1994), pp. 661–671.

[18] R. J. Vaccaro. “Lecture Notes on "Subspace-Based Signal Processing"”. Graduate School in Systems and Control, Katholieke Universiteit Leuven, Belgium. 2005.

[19] P. Van Overschee and B. De Moor. *Subspace identification for linear systems: Theory, implementation, applications*. Boston: Kluwer, 1996.

[20] J. C. Willems et al. “A note on persistency of excitation”. In: *Systems & Control Letters* 54.4 (2005), pp. 325–329. DOI: 10.1016/j.sysconle.2004.09.003.

[21] H. Zeiger and A. McEwen. “Approximate linear realizations of given dimension via Ho’s algorithm”. In: *IEEE Trans. Automat. Control* 19 (1974), pp. 153–153.