Invited session proposal "Low-rank approximation"

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Abstract

Low-rank approximations play an important role in systems theory and signal processing. The problems of model reduction and system identification can be posed and solved as a low-rank approximation problem for structured matrices. On the other hand, signal source separation, nonlinear system identification, and multidimensional signal processing and data analysis can be approached with powerful methods of low-rank tensor factorizations. The proposed invited session is focused on theoretical and algorithmic aspects of matrix and tensor low-rank approximations, with applications in system theory and multidimensional signal processing.

1 Introduction

The problem of approximating a matrix by a low-rank matrix has been extensively studied and well-understood. Low-rank approximations are widely used in signal processing, systems theory and machine learning as a tool for dimensionality reduction, feature extraction, and classification. The optimal solution can be obtained from the truncated singular value decomposition (SVD) [1]. In applications, however, more advanced (than the SVD) low-rank approximations are often needed, such as structured matrix low-rank approximations and tensor factorizations.

Structured low-rank approximations aim at approximating a matrix by a structured matrix of low-rank. Toeplitz, Hankel, and block-Hankel matrix structures are widely used in systems theory and signal processing (e.g., in system identification and model reduction). Low-rank approximation of Sylvester matrices is used for finding a greatest common divisor of polynomials approximately. Multivariate generalizations of Sylvester and Hankel matrices appear in problems in multidimensional systems theory and computations with multivariate polynomials. We refer to [2] for an overview of matrix structures and applications of structured low-rank approximation.

Tensor factorizations aim at approximating multidimensional data (tensors) by tensors with low-rank structure. Tensor decompositions are widely applied for analysis and processing of multidimensional data, e.g. for image processing and multi-way data analysis. In systems theory and signal processing, tensor factorizations are used in source separation and independent component analysis, and are gaining popularity in nonlinear system identification. For tensors, a variety of low-rank decompositions exist. *Multilinear* (or Tucker) decomposition is based on multilinear rank — a generalization of the notion of column and row rank of a matrix to tensors. *Canonical* (or PARAFAC) decomposition is the decomposition of a tensor in rank-1 terms (outer products of vectors). Block term decompositions were proposed as the unifying framework for the mentioned decompositions. Another two recent decompositions are the tensor train and hierarchical Tucker decompositions, meant for tensors of high orders. See [3, 4] for an overview of tensor decompositions and applications.

There are many challenges both in structured and tensor low-rank approximations. Firstly, for both problems the underlying optimization problem is, in general, non-convex and requires global optimization methods or effective relaxations techniques. Secondly, it is often challenging to handle specific constraints, such as structure, nonnegativity, and fixed and missing data in both low-rank matrix and tensor approximation. Finally, real-world applications pose computational challenges and require methods to handle large matrices and tensors.

2 Description of the proposal

The proposed invited session contains 11 talks, which represent state-of-the-art in the fields of structured low-rank approximations and tensor factorizations. We aim at gathering experts from both fields, in order to provide a unified view on the advances in these fields, which we believe is of interest to the audience of MTNS 2014. The invited session provides a wide coverage of theory and methods of matrix and tensor low-rank approximations, with applications in systems theory and multidimensional signal processing. The main aspects of the session and links between the talks are summarized below.

Theory

The main theoretical emphasis is made on the connections between matrix and tensor computations. These connections are especially important, since the corresponding communities are often considered to be separate. [Ish] shows equivalence between Tucker tensor decompositions and structured low-rank approximation. [DLa] considers theoretical properties of coupled matrix and tensor decomposition. [Use] considers simultaneous decomposition of symmetric tensors of different orders, and shows equivalence to matrix structured low-rank approximation.

Additionally, specific theoretical issues of matrix and tensor approximations are addressed. [Ste] focuses on the major problem of ill-posedness of the canonical decomposition of tensors. [GiZ] discusses the complexity of the structured low-rank approximation problem. [Sig] considers tensor decompositions in functional spaces. Both [Use] and [Shl] are closely connected to multidimensional systems theory.

Methods and algorithms

[Ose] gives an overview on recent advances of tensor decompositions, focusing on methods suitable for large-scale problems. [Mis] develops optimization methods on the manifold of low-rank semidefinite matrices, for solving algebraic Ricatti equations, which may be used for large-scale problems. [GiZ] develops a stochastic optimization algorithm for Hankel low-rank approximation, which can be also adapted to large-scale problems. [GiV] considers an efficient preconditioning technique for nonnegative matrix factorizations in the near-separable case.

Apart from large-scale methods, the following additional methods are considered. [DLa] develops methods for simultaneous decomposition of matrices and tensors. [Use] considers simultaneous decomposition of symmetric tensors of different orders (which is equivalent to a polynomial decomposition problem). [Pel] develops the convex technique of nuclear norm projection for Hankel matrices, in the problem of recursive identification.

Applications

A part of the papers consider examples of multidimensional signal processing problems. [Ose] gives an overview of problems from different branches of science, where tensor approximations are useful. **??** considers some problems of matrix approximation arising from the N-body problem. [GiV] considers challenging large-scale applications of hyperspectral image processing. [Shl] considers image processing problems for images with arbitrary shape.

Other papers deal with applications in systems and control. [GiZ] is meant for sum-of-damped exponentials modelling (autonomous system identification), [Pel] is aiming at input-output system identification. [Mis] aim at solving large-scale algebraic Riccatti equations. [Use] is motivated by a problem in nonlinear system identification.

3 Summary of the papers

[DLa] L. De Lathauwer, L. Sorber, M. Sorensen, and M. Van Barel, From tensor to coupled matrix/tensor decomposition

Summary: [DLa] discusses algorithms for joint decomposition of multiple matrices and tensors and its uniqueness properties. Applications can be found in recommender systems, advanced array processing systems and multimodal biomedical data analysis. The ideas are illustrated using Tensorlab, a recently released Matlab toolbox for tensor computations.

[GiZ] J. Gillard and A. Zhigljavsky, Stochastic methods for Hankel structured low rank approximation

Summary: [GiZ] discusses possible approaches for stochastic global optimization in the Hankel structured low-rank approximation problem. The authors propose a randomized Cadzow iterations algorithm with backtracking. This algorithm can be viewed as a random multistart-type algorithm but can be generalized so that it becomes an evolutionary method.

[GiV] N. Gillis and S. A. Vavasis, Semidefinite programming based preconditioning for more robust near-separable nonnegative matrix factorization

Summary: [GiV] discusses a problem of finding nonnegative matrix factorization in the near-separable case, which is motivated by problems in document classification an hyperspectral image processing. The authors propose a convex optimization (semidefinite programming) approach, that improves over state-of-the-art methods. Examples of large-scale hyperspectral images are considered.

[Ish] M. Ishteva and I. Markovsky, Tensor low multilinear rank approximation by structured matrix low-rank approximation

Summary: In [Ish], the tensor low-rank approximation problem in the sense of multilinear rank is reduced to a structured matrix low-rank problem. This allows dealing with weighted norms and missing and fixed elements in the given tensor. The case of structured (in particular, symmetric) tensors can be also considered in this framework.

[Mis] B. Mishra amd B. Vandereycken, A Riemannian approach to low-rank Riccati equation

Summary: [Mis] proposes a Riemannian optimization approach for computing low-rank solutions of the algebraic Riccati equation. The search space is the manifold of fixed-rank symmetric positive definite matrices. A Riemannian trust-region algorithm is proposed, which is scalable to large-scale problems.

[Ose] I. Oseledets, Numerical methods in higher dimensions using tensor factorizations

Summary: [Ose] discusses how the language of low-rank tensor factorization can provide a unified view on different algorithms for the solution of seemingly diverse and unconnected problems. Typical applications include quantum chemistry, chemical master equation in biology, and latent variable models in data analyses.

[Pel] K. Pelckmans, Low rank approximations in adaptive modeling approaches

Summary: [Pel] investigates the use of low nuclear norm matrices in methods of recursive identification and adaptive modelling. It is proposed to use a technique based on the nuclear-norm projection. Proper formulation leads to a strategy with high predictive power, while guaranteeing a low-dimensional minimal realization of the models which are built along the process.

[Shl] A. Shlemov and N. Golyandina, Shaped extensions of singular spectrum analysis

Summary: [Shl] deals with processing images of arbitrary shape (non-rectangular) by a new method (shaped singular spectrum analysis), which is based on low-rank approximation of quasi-Hankel matrices. Applications can be found in processing of gene expression data, seismic records, texture analysis, etc. The methods are applicable to large-scale data. An efficient implementation in the Rssa package for R scientific computing language is discussed.

[Sig] M. Signoretto and J. A. K. Suykens, Identification of structured dynamical systems in tensor product reproducing kernel Hilbert spaces

Summary: [Sig] combines techniques from multilinear algebra with statistical learning methods to derive structured models in reproducing kernel Hilbert spaces. The considered application domain is system identification, and in particular a class of discrete-time dynamical systems with input-output representation, which find applications in environmental modelling.

[Ste] A. Stegeman, Nonexistence of best low-rank approximations for real-valued three-way arrays and what to do about it

Summary: [Ste] discusses a way to deal with the cases when a best-fitting canonical decomposition of tensors does not exist. The main idea is to obtain a limit point of the canonical decomposition sequence featuring diverging components. An intuitive interpretation of the decomposition of the limit point is illustrated on a well-studied three-way dataset of ratings of TV shows.

[Use] K. Usevich, Decomposing multivariate polynomials with structured low-rank matrix completion

Summary: [Use] deals with decomposing multivariate polynomials into a sum of univariate polynomials of linear forms. This problem is equivalent to simultaneous decomposition of symmetric tensors of different orders. It is shown that the problem can be formulated as a structured low-rank low-rank matrix completion problem. A Matlab implementation is discussed.

References

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[3] T. G. Kolda and B. W. Bader. Tensor decompositions and applications. *SIAM Review*, 51(3):455–500, September 2009.

[4] L. Grasedyck, D. Kressner, and C. Tobler. A literature survey of low-rank tensor approximation techniques. *GAMM-Mitteilungen*, 36(1):53–78, 2013.