Some results on algebras with finite Gelfand-Kirillov dimension

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Introduction

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- **GK dimension** is a measure of the growth of an algebra.

- Specifically, we take a **finite dimensional subspace** $V$ of $\Lambda$ that contains $1$ and generates $\Lambda$ as an algebra and we look at how $\dim(V^n)$ grows as a function of $n$.

- If it grows like a **polynomial** of degree $d$, then we say that $\Lambda$ has **GK dimension** $d$. 

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- If $A$ has **GK dimension** 1 then $A$ is **commutative**.
Much is known on GK 0 and GK 1

- If $A$ has GK dimension 0 then $A = F$.

- If $A$ has GK dimension 1 then $A$ is commutative.

- But GK 2 is still mysterious...
Joint work with Jason Bell

We study finitely generated algebras of quadratic growth.

Given a field $k$ and a finitely generated $k$-algebra $A$, a $k$-subspace $V$ of $A$ is called a frame of $A$ if

- $V$ is finite dimensional,
- $1 \in V$,
- $V$ generates $A$ as a $k$-algebra.
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We say that $A$ has quadratic growth if there exist a frame $V$ of $A$ and constants $C_1, C_2 > 0$ such that

$$C_1 n^2 \leq \dim_k (V^n) \leq C_2 n^2$$

for all $n \geq 1$. 

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We note that an algebra of quadratic growth has GK dimension 2.

More generally, the GK dimension of a finitely generated \(k\)-algebra \(A\) is defined to be

\[
\text{GKdim}(A) = \limsup_{n \to \infty} \frac{\log \left( \dim(V^n) \right)}{\log n},
\]

where \(V\) is a frame of \(A\).
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If $A$ is a **finitely generated commutative algebra**, **GK dimension** is equal to **Krull dimension**.

**GK dimension** has seen great use over the years a useful tool for obtaining noncommutative analogues of results from **classical algebraic geometry**.
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If $A$ is a finitely generated commutative algebra, **GK dimension** is equal to **Krull dimension**.

**GK dimension** has seen great use over the years a useful tool for obtaining noncommutative analogues of results from classical algebraic geometry.

See **Krause and Lenagan** (2000).
Monomial algebras

We consider prime monomial algebras of quadratic growth.

A \( k \)-algebra \( A \) is a monomial algebra if

\[
A \cong k[x_1, \ldots, x_d]/I,
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for some ideal \( I \) generated by monomials in \( x_1, \ldots, x_d \).
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Monomial algebras are useful.

Why?

- Gröbner bases is the associated monomial algebra to a finitely generated algebra,
- many questions for algebras reduce to combinatorial problems for monomial algebras and can be studied in terms of forbidden subwords.
Joint work with Jason Bell

Theorem 1 Let

- $k$ be a field and
- let $A$ be a prime monomial $k$-algebra of quadratic growth.
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**Theorem 1** Let

- $k$ be a **field** and
- let $\mathcal{A}$ be a **prime monomial** $k$-algebra of **quadratic growth**.

Then the set of **primes** $P$ such that

$$\text{GKdim}(\mathcal{A}/P) = 1$$

is **finite**.

- Moreover, all such primes are **monomial ideals**.
- In particular, $\mathcal{A}$ has **bounded matrix images**.
Graded algebras of quadratic growth

Now we turn our attention to graded algebras of quadratic growth.

- We give an analogue of Bergman’s gap theorem.

- Bergman’s gap theorem states:

There are no algebras of GK dimension strictly between 1 and 2.
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Th. 2. Let $K$ be a field, $A = \bigoplus_{i=0}^{\infty} A_i$ affine graded $K$-algebra with quadratic growth, generated in degree $1$, $u \neq 0$ be a homogeneous element of $A$ and $(u)$ the ideal of $A$ generated by $u$. Then either:

- there is a natural number $m$ such that
  $$\dim_K \left( (u) \bigcap \bigoplus_{i=0}^{n} A_i \right) \geq \frac{(n - m)(n - m - 1)}{2}$$
  for all $n$ sufficiently large; or
- there is a positive constant $C$ such that for all $n$ sufficiently large
  $$\dim_K \left( (u) \bigcap \bigoplus_{i=1}^{n} A_i \right) < Cn.$$

Moreover, if $A$ is prime then the former holds.
How about prime ideals in graded algebras of quadratic growth?

**Th.3** Let $K$ be a field, and let $A = \bigoplus_{i=0}^{\infty} A_i$ be a prime affine graded non-PI $K$-algebra, generated in degree 1. If $A$ has quadratic growth then the intersection of all nonzero prime ideals $P$ such that $A/P$ has GK dimension 2 is nonzero, where we take an empty intersection to be all of $A$.

As a result of Th. 1, 3 we obtain the following corollary.

**Corollary 1.** Let $A$ be a finitely generated prime monomial algebra of quadratic growth. Then $A$ has bounded matrix images and either $A$ is primitive or has nonzero locally nilpotent Jacobson radical.
Remarks

- We note that a finitely generated prime monomial algebra that has GK dimension greater than 1 cannot be PI (see A. Belov, 1997).

- A conjecture of L. Small and J. Bell is that a finitely generated prime Noetherian algebra of quadratic growth is either primitive or PI.

- T. Gateva-Ivanova, E. Jespers and J. Okniński (2002) showed that Noetherian monomial algebras with finite GK dimension are PI.
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We show that prime monomial algebras $A$ of quadratic growth only have finitely many prime monomial ideals $P$ such that $A/P$ has GK dimension 1.

**Proposition 1.** Let $A$ be a prime monomial algebra of quadratic growth. Then $A$ has only finitely many prime monomial ideals $P$ such that $A/P$ has GK dimension 1.

**Proposition 2.** Let $k$ be a field and let $A$ be a prime monomial $k$-algebra of quadratic growth. Then every prime homomorphic image of $A$ of GK dimension 1 is also a monomial algebra.
Lemma 1 [L. Small] Let

\( A \) be a finitely generated prime graded algebra of quadratic growth,

and let \( Q \) be a nonzero prime ideal of \( A \) such that \( A/Q \) is finite dimensional.

Then either:

- \( Q \) is the maximal homogeneous ideal of \( A \),

- or \( Q \) contains a prime \( P \) such that \( A/P \) has GK dimension 1.
Lemma 2. Let $K$ be a field, let

$$A = \bigoplus_{i=0}^{\infty} A_i$$

be a graded prime $K$-algebra, and $Z$ be the extended centre of $A$. Suppose that $I$ is an ideal in $A$ that does not contain a nonzero homogeneous element and $z \in Z$, $x, y \in A$ are such that:

1. $x$ is a nonzero homogeneous element;
2. $y$ is a sum of homogeneous elements of degree smaller that the degree of $x$;
3. $x + y \in I$;
4. $zx = y$.

Then $z$ is not algebraic over $K$. 
Corollary 2. Let $K$ be a field and let

$$A = \bigoplus_{i=1}^{\infty} A_i$$

be a finitely generated prime graded non-PI $K$-algebra of quadratic growth.

If $P$ is a nonzero prime ideal of $A$ then either:

- $P$ is homogeneous
- or $A/P$ is PI.
We note that

- there do exist examples of prime monomial algebras of quadratic growth with nonzero Jacobson radical.

- If an algebra simply has GK dimension 2 and is not of quadratic growth then the collection of primes $P$ such that $A/P$ has GK dimension 2 can be VERY STRANGE;

  for example,

  they need not satisfy the ascending chain condition!
Open question

We pose the following question.

Trichotomy question

Let $\mathcal{A}$ be a prime finitely generated monomial algebra.

Is it true that $\mathcal{A}$ is either primitive, PI, or has nonzero Jacobson radical?

Any ideas are very welcome!!