Completeness and zero-dimensionality arising from the duality between closures and lattices

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Preface

Before beginning the mathematical part of this work, I would like to express my gratitude towards several persons, who helped me accomplish this research.

First of all my thanks go to E. Colebunders. As my supervisor she gave me all the opportunities to fulfill my dream of doing mathematical research in topology. She guided me into a wonderful world and always gave me valuable advice. The same holds for D. Aerts, who introduced me in the realm of the foundations of quantum mechanics and quantum logic.

I would also like to thank G.C.L. Brümmer, M. Erné and E. Giuli for their suggestions and tips concerning the theories involved in this thesis. I also thank S. Pulmannova for the support she gave me.

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Chapter 0

Introduction

Dualities between closure spaces or topological spaces and lattices have been a never-ending source of innovative approaches towards several problems. For instance the Stone representation theorem and the theory of frames and locales have had an important influence on modern topology [11].

In many other sciences, objects are modeled by means of lattices. For example, lattices are used by B. Ganter and R. Wille [29] in order to obtain a mathematical background for the analysis of formal contexts. Another example is the innovating approach towards quantum mechanics, initiated by G. Birkhoff and J. von Neumann [16] which lead to the foundations of quantum mechanics. In [5] lattices are used to introduce the concept of a state property systems where a physical entity is described by means of its states and a complete lattice representing its properties.

Frequently one associates to a lattice a closure operator on the underlying set. The closures which arise in this fashion are not necessarily topological closures. To be more precise they do not satisfy finite additivity, i.e. a union of two closed sets is not always closed. There are also numerous examples within mathematics of such non-additive closures. The well-known affine and vectorial closures from linear algebra and the convex closure are typical non-additive closures. The occurrences of such non-additive closures inside and outside of mathematics have lead topologists to study the category $\text{Cl}$ of closure spaces and continuous functions. Several equivalences between subcategories of $\text{Cl}$ and certain categories of complete lattices have been described by M. Erné [26], C.A. Faure [27] and others.
In this work our aim is to further analyze the different connections between lattices and closure spaces. We will focus on two particular properties of closure spaces: zero-dimensionality and completeness.

Zero-dimensionality is an important property since it appears throughout the examples and applications of closures spaces. For instance the convex closure is zero-dimensional, i.e. every convex set can be expressed as an intersection of convex sets, such that their complements are also convex. This unique property is often used in convex analysis. We will also show that zero-dimensionality provides a way of distinguishing between classical and quantum mechanical properties of a state property system. Finally we will use zero-dimensional closure spaces to present a version of the Stone representation theorem in the context of closure spaces, describing in more detail the duality [35] between certain zero-dimensional closure spaces and complemented partial orders.

Completeness is a well-known concept in categorical topology [18],[19]. In Top the search for complete objects yields the sober topological spaces and their representation by means of spatial frames. We will describe an analogue to this duality in the context of closure spaces. However as it shall appear, complete $T_0$ closure spaces differ substantially from sober topological spaces. We will be able to trace this divergence back to the fact that products in $\text{Cl}$ and $\text{Top}$ are formed differently.

When considering completeness in combination with zero-dimensionality for closure spaces, an obvious question is whether complete zero-dimensional closure spaces exist. We will show that, as for zero-dimensional topological spaces, there is no well-behaved notion of completeness in this context. However we will describe a uniform setting of non-Archimedean spaces, in which zero-dimensional closure spaces can be considered. For these non-Archimedean spaces we will develop a suitable completion theory. Moreover within this context we will find the classical completion of non-Archimedean uniform spaces [13] as a special case.

0.1 Overview

In the first chapter the well-fibred topological construct $\text{Cl}$ is introduced. As for topological spaces one can describe closure spaces by means of open subsets, closed subsets, a closure operator on a set or a family of neighborhood
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stacks. The $T_0$-objects of $\mathbf{Cl}$ yield the subcategory of $\mathbf{Cl}_0$ and the higher separation axioms $T_1$ and $T_2$ can also be defined. In a similar way as for topological spaces we use clopen sets in order to define zero-dimensionality and connectedness/disconnectedness for closure spaces. The full subcategory $0\mathbf{Cl}$ of zero-dimensional closure spaces is shown to be a bireflective subcategory of $\mathbf{Cl}$ and we prove that the subcategory $\mathbf{TDiscCl}$ of totally disconnected closure spaces is epireflective in $\mathbf{Cl}$. As for topological spaces the $\mathbf{TDiscCl}$-epireflection can be described as a quotient with respect to a connectedness relation $K$. One might think that the situation of closure spaces does not differ from the classical setting of topological spaces. However, the example of a finite totally disconnected Hausdorff closure space, which is not zero-dimensional, shows that this is not the case.

After we have introduced the elementary properties discussed above, we consider some applications of closure spaces. The first one is that of the well-known closures $\text{aff}$, $\text{vect}$ and $\text{conv}$ from linear algebra and convex analysis. We study the separation properties of these closures and we show that $\text{conv}$ is in fact a zero-dimensional Hausdorff closure operator. The next application is the closure operator introduced in [29] in order to describe “concepts” in formal context analysis. Finally our third example is the theory of state property systems. Here property lattices are used to model a physical entity. In [6] it was proven that the category $\mathbf{SP}$ of state property systems and the category $\mathbf{Cl}$ are equivalent. Hence with each state property system one associates a closure space (called the eigenclosure). Using this it was possible to show that certain physical properties of a state property system correspond with topological properties of the eigenclosure. For instance atomistic state property systems were shown to be equivalent with $T_1$ closure spaces in [47].

In Chapter 2 we use the equivalence between $\mathbf{SP}$ and $\mathbf{Cl}$ to further develop the theory of state property systems. Our objective will be to extract from a state property system information about its classical and non-classical (quantum mechanical) parts. Classicality of a state property system can be defined in several ways. For instance, an important difference between classical and quantum mechanical systems is that in the first case “superpositions” do not exist. Using the concept of superselection rules, we define s-classical state property systems as systems where there are no superpositions and we prove that the equivalence, introduced in the last part of Chapter 1, restricts to an equivalence between s-classical state property systems and topological spaces. Another way to look at classical properties is to consider deterministic properties which either hold or not (no quantum mechanical uncertainties). This leads to the concept
of a d-classical property. We show that such a property corresponds uniquely with a clopen subset of the associated eigenclosure. Using this result and the TDiscCl-epireflection of the eigenclosure we decompose a state property system into a number of pure non-classical state property systems (the connection components) and a totally classical one (the TDiscCl-epireflection). Finally the 0Cl-bireflection will enable us to extract from a state property system those properties which are superpositions of d-classical properties.

In the third chapter of this work we describe several representations of closure spaces using lattices and partial orders. First we develop a duality between the category of complete lattices with ∨, ⊤-preserving maps and a suitable full subcategory CCl₀ of Cl₀, similar to the duality between spatial frames and sober topological spaces [14]. The category CCl₀ is given by “complete” T₀ closure spaces, in which certain open based stacks must converge. The complete T₀ closure spaces can also be described as those closures for which every nonempty closed set is the closure of a unique point, hence they are also called point-closure spaces. We will see that the functors describing the mentioned duality are formally the same as the functors issued from the duality between spatial frames and sober topological spaces. Though there is a formal and categorical analogy between both dualities, one can not consider point-closure spaces as a generalization of sober topological spaces, since each Hausdorff topological space is sober but can never be a point-closure space.

Another series of representations we discuss is based on the theory developed in [26] by M. Erné. Using an invariant selection Σ on a lattice, an equivalence is obtained between the category LΣ of complete lattices L for which Σ(L) is a ∨-base and a full subcategory CΣ of Cl₀. We expand this equivalence in the following way. The invariant selection induces a canonical underlying functor UΣ : LΣ → Set, hence LΣ becomes a construct. Using the notion of concrete equivalence, introduced by H.E. Porst [40], we show that the equivalence described in [26] is a concrete equivalence when LΣ is considered as a construct. Afterwards we give three instances of this concrete equivalence. If Σ(L) = A(L) (the atoms of L) one obtains the concrete equivalence between atomistic lattices and T₁ closure spaces, which served as a starting point for the equivalence of atomistic state property systems and T₁ closure spaces [47]. If one chooses Σ(L) to be the set of ∨-irreducible elements of L, one obtains the equivalence between LΣ and the category of sober closure spaces, which are a generalization of sober topological spaces. Finally we consider Σ(L) = L \ {⊥} and find an equivalence between LΣ and the category of point-closure spaces CCl₀.

The last representation we present in Chapter 3 is a Stone-like theorem in
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In the context of closure spaces. First we observe that for a closure space $X$ the set of clopen subsets $CO(X)$ is not a boolean algebra but only a partial ordered set, together with a complementation. In [35] a duality between the category of complemented partial orders and a certain category of closure spaces is given. We show that this category is the full subcategory of $\text{Cl}$ consisting of zero-dimensional Hausdorff closure spaces which are compact in the sense that certain prime stacks must converge ($s$-compactness). The notion of $s$-compactness is not a generalization of topological compactness as is illustrated by a finite topological space which fails to be $s$-compact. The subcategory of $\text{0Cl}_0$ given by the $s$-compact spaces is epireflective and the epireflection is always an embedding, hence we do obtain a compactification.

In the fourth chapter we take a closer look at completeness. The search for a well-behaved categorical notion of completeness has lead G.C.L. Brümmer and E. Giuli to consider firm $U$-reflective subcategories, which can be viewed as consisting of complete objects. In the case of a complete well-powered construct $X$ and for the class $U_X$ of epimorphic embeddings, one has the existence of at most one firm $U_X$-reflective subcategory. Moreover if it exists, it is the full subcategory of $U_X$-injective objects and if $X$ is cogenerated by a class $P$ of $U_X$-injective objects, then the firm $U_X$-reflective subcategory is the epireflective hull $E_X(P)$.

We use these results to show that $\text{Cl}_0$, which is cogenerated by the Sierpinski space $S_2$, possesses a unique firm $U_{\text{Cl}_0}$-reflective subcategory. Using the regular closure ($b$-closure) of $\text{Cl}_0$, we show these complete objects to be the $b$-closed subspaces of powers of $S_2$ and prove that they are exactly the point-closure spaces. Hence it follows that $\text{CCl}_0$-spaces are indeed complete $T_0$ closure spaces. We also show that, as in the topological setting, the category $\text{0Cl}_0$ of zero-dimensional $T_0$ closure spaces does not have a firm $U_{\text{0Cl}_0}$-reflective subcategory.

In the last part of Chapter 3 we look at completeness in the context of a hereditary coreflective subcategory $C$ of a well-fibred topological construct $X$. We prove that in the case where the category $T_0X$ of $T_0$ objects is cogenerated by a class $P$ of $C$-objects, which are $U_{T_0X}$-injective, both categories $T_0X$ and $T_0C$ possess a subcategory of complete objects, which coincides with the epireflective hulls $E_{T_0X}(P)$ resp. $E_{T_0C}(P)$. To illustrate this theorem we construct a family of hereditary coreflective subcategories $\text{Tight}(\alpha)$ of $\text{Cl}$. In this case the complete objects of $\text{Tight}_0(\alpha)$ are given by the $b$-closed subspaces of powers of $S_2$, where the powers are taken in $\text{Tight}_0(\alpha)$. Another example is the category $\text{Top}$, which is a hereditary coreflective subcategory of $\text{Cl}$. Both $\text{Top}_0$ and $\text{Cl}_0$...
have subcategories of complete objects (\(\text{Sob} \ \text{resp.} \ \text{CCl}_0\)). In both cases the complete objects are \(b\)-closed subspaces of powers of \(S_m\), the difference between them is due to the fact that the products in \(\text{Top}\) and \(\text{Cl}\) differ.

Since \(0\text{Cl}_0\) does not have a firm \(\mathcal{U}_{0\text{Cl}_0}\)-reflective subcategory of complete objects and because these spaces have been shown to be useful, for instance in the context of the Stone representation theorem for closure spaces, we would like to have some notion of completeness for these spaces. To achieve this we will look at a uniform setting where these spaces can be considered, such that one can formulate a completion theory.

Our starting point is H. Herrlich’s theory of (pre-)nearness spaces [31]. In Chapter 5 we introduce the category \(\text{UPNear}\) of uniform pre-nearness spaces. We give equivalent descriptions for these spaces using uniform covers, small collections, near collections, entourages and families of pseudometrics. We show that \(\text{Unif}\) is a coreflective subcategory of \(\text{UPNear}\) and that there is a canonical forgetful functor from \(\text{UPNear}\) to \(\text{Cl}\), hence we can consider the problem of uniformizability of closure spaces. As for topological spaces the uniformizable closure spaces are exactly the completely regular ones. A special class of completely regular closures are the zero-dimensional ones. This fact will lead to the notion of non-Archimedean spaces in the last chapter of this thesis.

In [53] a representation of pre-nearness spaces is described by means of an adjunction with the category \(\text{PGrL}\) of pre-grill lattices. To conclude the fifth chapter we show that this adjunction can be restricted to an adjunction between \(\text{UPNear}\) and the category \(\text{UPGrL}\) of uniform pre-grill lattices.

In the last chapter we introduce and study the category \(\text{NA}\) of non-Archimedean spaces. These spaces are uniform pre-nearness spaces of which the structure is entirely determined by means of uniform partitions, equivalence relations or ultra-pseudometrics. The canonical closure space induced by a non-Archimedean space is zero-dimensional and conversely every zero-dimensional closure space is non-Archimedean uniformizable in a canonical way.

We show that the representation we mentioned earlier of \(\text{UPNear}\) by means of \(\text{UPGrL}\), can be restricted in the non-Archimedean setting to an adjunction between \(\text{NA}\) and the category of non-Archimedean pre-grill lattices.

Finally we turn to the aspect of completeness for non-Archimedean spaces. We show that the category \(\text{NA}_0\) (the \(T_0\)-objects of \(\text{NA}\)) has a unique firm \(\mathcal{U}_{\text{NA}_0}\)-reflective subcategory of complete objects. Because, for a non-Archimedean
space $X$, the refinement relation $<$ on the collection $\nu X$ of uniform partitions forms a partial order, we are able to characterize the complete non-Archimedean spaces using order-preserving choice functions. Such an order-preserving choice function is a map $f : \nu X \to \cup \nu X$ which associates to every uniform partition $\mathcal{P} \in \nu X$ one of its members $f(\mathcal{P}) \in \mathcal{P}$, in such a way that $\mathcal{P} < \mathcal{Q}$ implies $f(\mathcal{P}) \subset f(\mathcal{Q})$. A non-Archimedean space is complete if and only if every order preserving function is convergent, i.e. $\cap_{\mathcal{P} \in \nu X} f(\mathcal{P}) \neq \emptyset$. Since the category of non-Archimedean uniform spaces is a hereditary coreflective subcategory of $\text{NA}$, we can show that the completion of a non-Archimedean Hausdorff uniform space in $\text{NA}$ reduces to the classical completion. Finally we also show that completeness and (totally) boundedness of a non-Archimedean space is not equivalent to $s$-compactness of the underlying zero-dimensional closure space.

0.2 Notations

Throughout this work we will frequently use the following notations:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A, B, C, X, Y, Z, \ldots$</td>
<td>sets</td>
</tr>
<tr>
<td>$a, b, c, x, y, z, \ldots$</td>
<td>points</td>
</tr>
<tr>
<td>$\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{U}, \ldots$</td>
<td>sets of sets</td>
</tr>
<tr>
<td>$\alpha, \beta, \gamma, \mu, \ldots$</td>
<td>sets of sets of sets</td>
</tr>
<tr>
<td>$\mathcal{L}, \mathcal{M}, \mathcal{K}, \ldots$</td>
<td>lattices</td>
</tr>
<tr>
<td>$\text{Top}, \text{Unif}, \mathcal{A}, \mathcal{B}, \ldots$</td>
<td>categories</td>
</tr>
<tr>
<td>$X, Y, \ldots$</td>
<td>objects</td>
</tr>
<tr>
<td>$f, g, \ldots$</td>
<td>morphisms</td>
</tr>
<tr>
<td>$F, G, \ldots$</td>
<td>functors</td>
</tr>
<tr>
<td>$\epsilon, \eta, \ldots$</td>
<td>natural transformations</td>
</tr>
<tr>
<td>$\psi, \varphi$</td>
<td>pseudometrics</td>
</tr>
<tr>
<td>$\Psi, \Phi$</td>
<td>collections of pseudometrics</td>
</tr>
<tr>
<td>$\mathcal{P}(X)$</td>
<td>powerset of a set $X$</td>
</tr>
<tr>
<td>$\mathcal{P}^2(X)$</td>
<td>powerset of the powerset of a set $X$</td>
</tr>
</tbody>
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Chapter 1

Closure spaces

1.1 Elementary definitions and results

In many disciplines of mathematics, in quantum physics, in formal context analysis and in many other sciences, closure operators arise in a natural fashion as we will see in the examples at the end of this chapter and in the next one. However such closures do not always behave as topological closures, often they are not additive. Closure spaces [23], [24] are a generalization of topological spaces and are introduced in order to capture these examples and to apply topological methods in these situations. They arise if one drops the axiom stating that a finite union of closed sets is again closed in the definition of topological spaces.

We now formally introduce the concept of a closure space and discuss some of their elementary properties.

**Definition 1.1.1.** A **closure space** is a couple \((X, \mathcal{C})\) where \(X\) is a set and \(\mathcal{C}\) a set of subsets of \(X\) which contains the empty set and which is closed for arbitrary intersections. A set in \(\mathcal{C}\) will be called a **closed subset** of \(X\). A subset \(B\) of \(\mathcal{C}\) will be called a **base for closed sets** if each set in \(\mathcal{C}\) can be obtained as an intersection of elements of \(B\). \(\mathcal{C}\) is then said to be generated by \(B\).

**Example 1.1.2.** [The Sierpinski space \(S_2\) and a closure space which is not topological.]

From the definition it is clear that any topological space is a closure space, so the **Sierpinski space** \(S_2 = (\{0, 1\}, \{\emptyset, \{0\}, \{0, 1\}\})\) is a closure space. However the space given by \((X = \{0, 1, 2\}, \mathcal{C} = \{\emptyset, X, \{0\}, \{1\}\})\) is a closure space but not a topological space, since \(\{0, 1\}\) is not closed.
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As for topological spaces one can characterize a closure space in several other ways instead of using closed subsets. The proofs of the subsequent propositions are completely analogous to the ones given in the topological case, we refer the reader to any book on general topology for those.

**Proposition 1.1.3.** Let \((X, \mathcal{C})\) be a closure space, define the **closure of a set** \(M \subset X\) as follows \(\text{cl}(M) = \cap\{C \in \mathcal{C} | M \subset C\}\). This closure operation has the following properties:

1. \(\text{cl}(\emptyset) = \emptyset\) (*groundedness*)
2. \(\forall A \subset X : A \subset \text{cl}(A)\) (*extensiveness*)
3. \(A \subset B \subset X \Rightarrow \text{cl}(A) \subset \text{cl}(B)\) (*monotonicity*)
4. \(\text{cl}(\text{cl}(A)) = \text{cl}(A)\) (*idempotency*)

Conversely every operator \(\text{cl} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)\) on a set \(X\) satisfying these four properties defines a unique closure space \((X, \mathcal{C})\) by

\[
\mathcal{C} = \{A \in \mathcal{P}(X)|\text{cl}(A) = A\}
\]

**Remark 1.1.4.** Some authors ([26], [27], ... ) define closure spaces as above, but without asking that the closure is grounded, i.e. \(\emptyset\) needs not be closed. We however, want to have a unique closure structure on each set of cardinality 1, in order to obtain a well-fibred topological category in the sense of [1].

Another equivalent way to describe closure spaces is using the notion of open subsets.

**Proposition 1.1.5.** Let \((X, \mathcal{C})\) be a closure space. A subset \(A\) is called an **open subset** if its complement is closed. The collection \(\mathcal{O}\) of open subsets always contains \(X\) and is closed for arbitrary unions. Conversely, every family \(\mathcal{O}\) of subsets of \(X\), satisfying the above conditions induces a unique closure space by complementation. A subset \(B\) of \(\mathcal{O}\) is a **base for open sets** if each set in \(\mathcal{O}\) can be obtained as an union of elements of \(B\), \(\mathcal{O}\) is then said to be generated by \(B\).

Finally a last way in which we will describe closure spaces is by means of neighborhood stacks. We recall that for any set \(X\) and \(A \subset \mathcal{P}(X)\), we define as usual stack \(\mathcal{A} = \{B \subset X | \exists A \in A : A \subset B\}\). A collection \(A\) such that \(A = \text{stack } A\) is called a **stack**. A proper stack is a non-empty stack \(A\) such that \(\emptyset \notin A\). A proper stack \(A\) in a closure space is said to be an **open based stack** if \(A = \text{stack } \{G \in A | G \text{ open}\}\).
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**Proposition 1.1.6.** Let \((X, C)\) be a closure space with \(O\) as a collection of open sets, then we write \(V(x) = \text{stack } \{A \in O | x \in A\}\), for any \(x \in X\). We call \(V(x)\) the **neighborhood stack** of \(x\). The collection \((V(x))_{x \in X}\) has the following properties:

1. \(X \in V(x)\)
2. \(\forall V \in V(x) : x \in V\)
3. \(V \in V(x), V \subset W \Rightarrow W \in V(x)\)
4. \(\forall V \in V(x) : \exists W \in V(x) : \forall y \in W : V \in V(y)\)

Conversely any collection \((V(x))_{x \in X}\) satisfying the above will induce a unique closure space structure on \(X\) with the open sets given by:

\[ O = \{A \subset X | \forall x \in A : A \in V(x)\} \]

Clearly in a closure space \((X, C)\) every neighborhood stack \(V(x)\) is an open based stack. As in the topological case we have for the closure operator \(\text{cl}\) associated with \((X, C): x \in \text{cl}(A)\) if and only if \(V \cap A \neq \emptyset\) for every \(V \in V(x)\).

**Remark 1.1.7.** At this point we note that these characterizations are parallel with the usual topological ones. However an important difference, which will have quite a few consequences when we’ll be looking at completeness, is the fact that the neighborhood stacks are not necessarily filters in the case of closure spaces.

We now have several different characterizations for closure spaces at our disposal. We shall write \(X\) for a closure space, the underlying set will be written as \(X\) and we’ll use \(C_X\) for the collection of closed subsets, \(O_X\) for the collection of open subsets, \(d[X]\) for the closure operation and \(V_X(x)\) for the neighborhood stack of a point \(x \in X\).

We will now turn our attention to the concept of continuity.

**Definition 1.1.8.** A morphism \(f : X \to Y\) between two closure spaces, is a function \(f : X \to Y\), such that:

\[
\forall C \in C_Y : f^{-1}(C) \in C_X
\]

\(f\) is then called a **continuous function**.
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**Proposition 1.1.9.** Let $X$ and $Y$ be closure spaces and let $f : X \rightarrow Y$ be a function. The following are equivalent:

1. $f$ is continuous
2. $\forall O \in \mathcal{O}_Y : f^{-1}(O) \in \mathcal{O}_X$
3. $\forall A \subset X : f(\text{cl}_X(A)) \subset \text{cl}_Y(f(A))$
4. $\forall x \in X : \forall W \in \mathcal{V}_Y(f(x)) : f^{-1}(W) \in \mathcal{V}_X(x)$

Now that we have both objects and morphisms we can form the category of closure spaces which we shall write $\text{Cl}$.

**Definition 1.1.10.** [1] Let $X$ be a concrete category with forgetful functor $U : X \rightarrow A$. It is a topological category provided that $U$ is a topological functor, i.e. for every $U$-structured source $(f_i : X \rightarrow U\mathcal{X}_i)_{i \in I}$ there exists a unique initial lift $(\bar{f}_i : X \rightarrow \mathcal{X}_i)_{i \in I}$. If $X$ is concrete over $\text{Set}$, it will be called a construct, in this case we will sometimes write $X$ for the underlying set $U\mathcal{X}$ of an object $\mathcal{X}$. For a construct $X$ will write $\text{Emb} X$ for the class of all embeddings of $X$ and $\text{Epi} X$ for the class of all its epimorphisms. An $\text{Emb} X$-subobject will be sometimes referred to as a subspace. A subcategory $C$ of $X$ which is closed under the formation of $\text{Emb} X$-subobjects will be called a hereditary subcategory. A construct is well-fibred if it is fibre-small and for every set with at most one element the fibre has exactly one element.

**Remark 1.1.11.** Note that a well-fibred topological construct is a topological category in the sense of [43].

With the obvious forgetful functor $U : \text{Cl} \rightarrow \text{Set}$, $\text{Cl}$ becomes a construct. As for $\text{Top}$, we know that $\text{Cl}$ is in fact a well-fibred topological construct. The initial lift of a $U$-structured source $(f_i : X \rightarrow U\mathcal{X}_i)_{i \in I}$, is given by $(\bar{f}_i : X \rightarrow \mathcal{X}_i)_{i \in I}$. Here the closed sets of $\mathcal{X}_m$ are given by

$$C_m = \{ \cap_{j \in J} f_j^{-1}(C_j) | C_j \in \mathcal{C}_{\mathcal{X}_j}, J \subset I \}$$

i.e. $\{ f_i^{-1}(C_i) | C_i \in \mathcal{C}_{\mathcal{X}_i}, i \in I \}$ is a base for $C_m$. Equivalently we have for the open sets:

$$O_m = \{ \cup_{j \in J} f_j^{-1}(A_j) | A_j \in \mathcal{O}_{\mathcal{X}_j}, J \subset I \}$$

A special case of initial structures is of course the product closure space $\Pi_{i \in I} X_i$ of a family $(X_i)_{i \in I}$, a base for the closed sets of $\Pi_{i \in I} X_i$ is given by
1. Closure spaces

\{pr_i^{-1}(C_i)|C_i \in C_X, i \in I\}, where \(pr_i\) denotes the \(i\)th projection. **Subspaces of a closure space** are made in the usual way: for a subset \(A \subset X\) of a closure space \(X\) we take the initial lift of the canonical inclusion \(i : A \to UX\), the closed sets are given by \(C_A = \{C \cap A | C \in C_X\}\). As a topological construct \(\mathbf{Cl}\) has discrete and indiscrete objects: the **discrete closure space** with underlying set \(X\) is \(D_X\) with closed sets \(P(X)\) and the **indiscrete closure space** is \(I_X\), where the only closed sets are \(\emptyset\) and \(X\). In \(\mathbf{Cl}\) the epimorphisms are exactly the surjective continuous maps and the monomorphisms are the injective ones. The extremal epimorphisms of \(\mathbf{Cl}\) are the quotients and the extremal monomorphisms are the embeddings.

**Remark 1.1.12.** One of the main differences between topological spaces and closure spaces, is the formation of products. For example in \(\mathbf{Top}\) the product of two discrete spaces is again a discrete space. In \(\mathbf{Cl}\) this is not the case. This fundamental difference in the products of both categories will have quite an impact later on.

We know that any topological space is a closure space, moreover any map between topological spaces is continuous in the topological sense if and only if it is continuous in the sense of closure spaces. From this it follows that \(\mathbf{Top}\) is a full subcategory of \(\mathbf{Cl}\). In fact it is a bireflective subcategory of \(\mathbf{Cl}\) [24]. The bireflection of a closure space \(X\) is the topological space on \(X\) for which \(C_X\) forms a sub-basis for closed sets. For more concerning general category theory, topological categories and subcategories we refer to [1], [43], [31]. We refer to [24], [23], [25] for more about closure operators, closure spaces and the category \(\mathbf{Cl}\).

### 1.2 Properties of closure spaces

Now that we have had a first encounter with the concept of closure spaces, we will introduce some of the properties such spaces can have. We will focus on separation properties (\(T_0\), \(T_1\) and \(T_2\)), disconnectedness and zero-dimensionality of closure spaces. The definitions we will use formally coincide with the usual definitions in a topological context, however they have their roots in the various applications of closure spaces as will be shown in later examples. For instance the \(T_0\) and \(T_1\) properties correspond to the concept of state determination and atomisticity of property lattices in quantum logic. For more details about separation axioms for closure spaces and their applications we refer to [51], [6] or
1. Closure spaces

[47]. Later on in this thesis we will pay some special attention to so called non-Archimedean spaces. For this we will need to generalize the classical concepts of zero-dimensionality and disconnectedness to closure spaces. We will show that these generalized properties are exactly the ones which occur naturally in the context of quantum physics and convex analysis.

1.2.1 Separation properties

Definition 1.2.1. A closure space $X$ is said to be a $T_0$ closure space if for any two different points $x, y$ of $X$:

$$\exists V \in \mathcal{V}_X(x) : y \notin V \text{ or } \exists V \in \mathcal{V}_X(y) : x \notin V$$

Another way to characterize the $T_0$ property is the following.

Proposition 1.2.2. A closure space $X$ is $T_0$ if and only if for every two distinct points $x, y$ one has $\text{cl}_X(\{x\}) \neq \text{cl}_X(\{y\})$.

These spaces can also be characterized in a more categorical way as follows from the next proposition, which can be proven in the same way as its counterpart in the topological case.

Proposition 1.2.3. A closure space $X$ is a $T_0$ closure space if and only if every $\text{Cl}$-morphism $f : L_2 \to X$ is constant.

This proposition shows that the $T_0$ closure spaces are exactly the $T_0$-objects of the category $\text{Cl}$ as defined by T. Marny in [34].

Definition 1.2.4. A $T_0$-object of a well-fibred topological construct $X$, is an object $\overline{X}$ such that every $X$-morphism $f : L_2 \to \overline{X}$ is constant.

Definition 1.2.5. A closure space $X$ is said to be a $T_1$ closure space if for any two different points $x, y$ of $X$:

$$\exists V \in \mathcal{V}_X(x) : y \notin V \text{ and } \exists V \in \mathcal{V}_X(y) : x \notin V$$
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As for topological spaces this can be reformulated as follows.

**Proposition 1.2.6.** A closure space is a $T_1$ closure space if and only if every singleton is closed.

The full subcategory of $\text{Cl}$, given by the $T_1$ closure spaces will be written as $\text{Cl}_1$. This subcategory is extremally epireflective in $\text{Cl}$, as can be found in [48].

**Definition 1.2.7.** A closure space $X$ is said to be a $T_2$ closure space or Hausdorff closure space if for any two different points $x, y$ of $X$:

$$\exists V \in \mathcal{V}_X(x) : \exists W \in \mathcal{V}_X(y) : V \cap W = \emptyset$$

The Hausdorff closure spaces induce an extremally epireflective full subcategory of $\text{Cl}$ [48], which we will write as $\text{Cl}_2$. As for topological spaces we have the following implications:

$$T_2 \Rightarrow T_1 \Rightarrow T_0$$

### 1.2.2 Disconnectedness and zero-dimensionality

**Definition 1.2.8.** A subset $C \subset X$ in a closure space $X$ is called a clopen subset if both $C$ and its complement are closed.

**Definition 1.2.9.** A closure space is called a zero-dimensional closure space if it has a base for the closed sets, consisting only of clopen sets, or equivalently, a base for the open sets consisting of clopen sets.

As in the topological case we have the following characterization.

**Proposition 1.2.10.** A closure space is zero-dimensional if and only if every neighborhood stack has a base of clopen sets.

**Corollary 1.2.11.** Every $T_0$ zero-dimensional closure space is Hausdorff.

It is clear that zero-dimensionality is productive and inherited by subspaces in $\text{Cl}$, moreover any indiscrete closure space is zero-dimensional. Hence the full subcategory $\text{0Cl}$ of zero-dimensional closure spaces is a bireflective subcategory of $\text{Cl}$. The $\text{0Cl}$-bireflection of a closure space $X$ is given by the closure structure of which the open subsets are generated by the collection $\text{CO}(X)$ of clopen sets of $X$.
Definition 1.2.12. A closure space $X$ is called a **connected closure space** if there are no clopen sets, other than $\emptyset$ and $X$. It is called a **disconnected closure space** if it is not connected and a **totally disconnected closure space** if all non-empty, connected subspaces contain only one point.

A subset $A \subset X$ of a closure space $X$ is called connected if the induced subspace $\overline{A}$ is connected, the same holds for disconnected and totally disconnected subsets. As in the case of a topological space we have the following.

**Proposition 1.2.13.** Every zero-dimensional $T_0$ closure space is totally disconnected.

**Proposition 1.2.14.** Let $X$ be a closure space.

1. $X$ is connected if and only if every $\text{Cl}$-morphism $f : X \to \mathcal{D}_2$ is constant.
2. $X$ is totally disconnected if and only if every $\text{Cl}$-morphism $f : Y \to X$ from a connected closure space $Y$ is constant.

We place these concepts in a categorical setting, recalling some definitions of [10], [41] and [42]. In a well-fibred topological construct $C$ a $(\mathcal{P}-)$ **connectedness** is a class

$$\mathcal{C}_\mathcal{P} = \{X \in |C||\text{every } C\text{-morphism } f : X \to P \text{ with codomain in } P \text{ is constant}\}$$

and a $(\mathcal{Q}-)$ **disconnectedness** is a class

$$\mathcal{D}_\mathcal{Q} = \{X \in |C||\text{every } C\text{-morphism } f : Q \to X \text{ with domain in } Q \text{ is constant}\}$$

(here $\mathcal{P}$ and $\mathcal{Q}$ are isomorphism closed classes of $C$-objects).

From the previous proposition we see that the class of all connected closure spaces is a connectedness $\mathcal{C}_\mathcal{D}_2$ and that the class of all totally disconnected closure spaces form the disconnectedness $\mathcal{D}_\mathcal{C}_\mathcal{D}_2$, where $\mathcal{D}_2$ is the class all two-point discrete spaces. The full subcategories induced by these classes will be written as $\text{ConnCl}$ and $\text{TDiscCl}$. As a disconnectedness of a well-fibred topological construct $\text{TDiscCl}$ is an extremally epireflective subcategory of $\text{Cl}$. Since we will use this extremal epireflection later on in a more physical setting, in order to separate classical and quantum properties of a physical entity, we give a more explicit description. In fact the $\text{TDiscCl}$ extremal epireflection is analogous to the construction in the topological case, however some of the proofs must be altered to suit the context of closure spaces.

The first lemma is proven in the same way as in the topological case.
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Lemma 1.2.15. Let $\mathcal{A}$ be a chained collection of connected subsets of a closure space $X$. Then $\bigcup \mathcal{A}$ is also connected.

The usual proof of the next lemma in the topological case involves the fact that two continuous functions $f, g : X \to Y$ from a topological space to a Hausdorff topological space, coinciding on a dense subset of $X$ must be equal. This fact however is not true in the case of closure spaces as is shown by the next example.

Example 1.2.16. [Two Cl-morphisms to a Hausdorff closure space, coinciding on a dense subset which are different.]

Let $X$ be the closure space on $X = \{1, 2, 3\}$ with closed sets $C_X = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\}$. Let $A$ be the dense subset $\{2, 3\}$ and consider the morphisms $f, g : X \to D_2$ given by $f(2) = 1, f(3) = 0$ and $g(2) = 1, g(3) = 0$ and $f(1) = 0, g(1) = 1$. Then $f, g$ coincide on $A$ but are different.

Lemma 1.2.17. Whenever $A \subset X$ is connected in a closure space $X$, then $\text{cl}_X(A)$ is also connected.

Proof. Let $B$ be a clopen subset in $(\text{cl}_X(A), C_{\text{cl}_X(A)})$, then there are $C_1, C_2 \in C_X$ such that $B = \text{cl}_X(A) \cap C_1$ and $\text{cl}_X(A) \setminus B = \text{cl}_X(A) \cap C_2$, hence both $B$ and $\text{cl}_X(A) \setminus B$ are closed in $X$. Since $A \subset \text{cl}_X(A)$ we have that $B \cap A$ is clopen in the subspace $A$ and since this is a connected space one has either $A \cap B = A$ or $A \cap B = \emptyset$. In the first case it follows that $A \subset B \subset \text{cl}_X(A)$, and by the fact that $B$ is a closed subset of $X$ we get that $\text{cl}_X(A) = B$. In the second case $A \subset \text{cl}_X(A) \setminus B \subset \text{cl}_X(A)$, hence $\text{cl}_X(A) = \text{cl}_X(A) \setminus B$ so $B = \emptyset$. □

From these two lemma’s the following proposition can be obtained easily.

Proposition 1.2.18. For each point $x \in X$ of a closure space $X$ there exists a largest connected subset $K(x) = \bigcup\{A \subset X | x \in A, A \text{ connected}\}$ containing $x$. Moreover $K(x)$ is always closed in $X$ and $\{K(x) | x \in X\}$ is a partition of $X$, hence it defines an equivalence relation: $xKy \Leftrightarrow K(x) = K(y)$.

Definition 1.2.19. For a closure space $X$ we will call $K(x)$ the connection component of $x \in X$ and $K$ the connection relation of $(X, C)$.

Corollary 1.2.20. For a totally disconnected closure space $X$ we have that for every $x \in X$: $K(x) = \{x\}$. Since this is a closed set we know that $X$ is T₁.

Proposition 1.2.21. Let $X$ be a closure space. The quotient

$$q : X \to \text{cl}_X : x \mapsto K(x)$$

is the TDiscCl extremal epireflection on $X$ in Cl.
1. Closure spaces

Proof. Consider the quotient space $\overline{X}_{|K} = (X_{|K}, C_{|K})$ and suppose that there is a closed subspace $A \subset X_{|K}$ containing two different points $K(x)$ and $K(y)$. Then $B = q^{-1}(A)$ is a closed subset of $\overline{X}$ which is not connected since it contains both $K(x)$ and $K(y)$, hence there is a proper clopen subset $C$ of $B$. There are $C_1, C_2 \in C_X$ such that $C = B \cap C_1$ and $B \setminus C = B \cap C_2$, so $C$ and $B \setminus C$ are also closed in $\overline{X}$ and whenever $z \in C$, we have that $K(z) \subset C$. Setting $D = q(C)$ we get a proper subset of $A$, moreover since $q$ is a quotient and $q^{-1}(D) = C$ and $q^{-1}(A \setminus D) = B \setminus C$ are closed in $\overline{X}$, $D$ is clopen in $A$, hence $A$ is not connected.

To prove that $q$ is an extremal epireflection we take a morphism $f : X \to Y$ with $Y$ a totally disconnected closure space. It’s easy to verify that if $A$ is a connected subset of $X$, then $f(A)$ is connected in $Y$, hence if $x K y$ then $f(x) K f(y)$. Since $Y$ is totally disconnected $f(x) = f(y)$. This allows us to define for each $f : X \to Y$ a unique map $g : \overline{X}_{|K} \to Y : K(x) \mapsto f(x)$. Since for any closed set $A$ of $Y$ one has that $q^{-1}(g^{-1}(A)) = q^{-1}(\{K(a) | f(a) \in A\}) = f^{-1}(A)$ is also closed, the map $g$ is a Cl-morphism. For this morphism we have the factorization we were looking for:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
| & \downarrow{q} & \\
\overline{X}_{|K} & \xrightarrow{g} & 
\end{array}
$$

From the preceding result, it might appear that closure spaces behave in the same way as topological spaces. However, the next examples show that there are fundamental differences between Top and Cl.

Example 1.2.22. [A finite Hausdorff closure space which is not discrete. Moreover it is a zero-dimensional totally disconnected closure space.]

Let $X = \{a, b, c\}$ and consider the closure space $\overline{X}$ generated by the base for open sets $\{\{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ (see Figure 1.1). This is the space used on the cover of this thesis, where it is depicted together with its lattice of open subsets. It is not a topological space since the open sets are

$$O_{\overline{X}} = \{X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \emptyset, \{b, c\}\}$$

and $\{a\}$ is not open. It is easily verified that $\overline{X}$ is a zero-dimensional closure space which is totally disconnected, hence $T_1$. By the existence of clopen neighborhoods one has that it is a finite Hausdorff space. Remark that in the topological case a finite Hausdorff space can only be a discrete space.
1. Closure spaces

![Figure 1.1: The basic open sets of the space from Example 1.2.22.](image1)

**Example 1.2.23.** [A closure space which is finite, totally disconnected and Hausdorff but not zero-dimensional.]

The relation between totally disconnectedness and zero-dimensionality for closure spaces is not completely identical to the situation for topological spaces. Let $X = \{1, 2, 3, 4\}$ and choose as a base for the open sets of a closure space $X$ the following: $\{\{1\}, \{2\}, \{1, 4\}, \{1, 3\}, \{2, 3\}, \{2, 3, 4\}\}$ (see Figure 1.2). We have:

$\mathcal{O}_X = \{\{1\}, \{2\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3, 4\}, \emptyset\}$

$\mathcal{C}_X = \{\{1\}, \{2\}, \{3\}, \{4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{2, 4\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3, 4\}, \emptyset\}$

From this we see that the proper clopen sets are

$\{\{2\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1\}, \{1, 4\}, \{2, 3\}\}$

However they do not form a base since $\{1, 3\}$ is not a union of clopen sets, hence the closure space $X$ is not zero-dimensional. It is however Hausdorff.
Moreover it is also a totally disconnected closure space, hence we have a finite totally disconnected, Hausdorff space. In the topological case such a space is always zero-dimensional, this does not hold for closure spaces as is shown by this example.

1.3 Applications of closure spaces

To finish this chapter we give a few examples of applications of closure spaces. We will treat closure spaces in the context of linear algebra and convex analysis. We will also see that closure spaces arise spontaneously in formal context analysis and finally we give an example from quantum logic, which we shall use in the next chapter to illustrate an important application of disconnected and zero-dimensional closure spaces. This list is certainly not exhaustive, many other application of closures exists (geometry [28], social sciences [12], ...) but it gives a nice overview of the diversity of fields where closure spaces are encountered.

1.3.1 In linear algebra and convex analysis

Our first series of applications of the concept of closure spaces is given by the following well-known closures from linear algebra and convex analysis.

Consider a real vector space $E^n$ and define for $S \subset E^n$ the following:

$$vect(S) = \{ \sum_{i=0}^{n} \lambda_i \vec{s}_i | n \in \mathbb{N}, \forall 0 \leq i \leq n : \vec{s}_i \in S, \lambda_i \in \mathbb{R} \}$$

$$aff(S) = \{ \sum_{i=0}^{n} \lambda_i \vec{s}_i | n \in \mathbb{N}, \forall 0 \leq i \leq n : \vec{s}_i \in S, \lambda_i \in \mathbb{R}, \sum_{i=0}^{n} \lambda_i = 1 \}$$

$$conv(S) = \{ \sum_{i=0}^{n} \lambda_i \vec{s}_i | n \in \mathbb{N}, \forall 0 \leq i \leq n : \vec{s}_i \in S, \lambda_i \in \mathbb{R}^+, \sum_{i=0}^{n} \lambda_i = 1 \}$$

Each of these operations is a closure, hence $(E^n, vect)$, $(E^n, aff)$ and $(E^n, conv)$ are closure spaces. Moreover these closure operations are not additive, hence these closure spaces are not topological. The respective closed subsets of these spaces are the subvectorspaces of $E^n$ and the empty set, the affine sets and the convex sets of $E^n$.

Remark 1.3.1. We note here that in linear algebra for algebraic reasons one defines $vect(\emptyset) = \{ \vec{0} \}$. Since we will be developing our results for grounded closure spaces, we will use the above notion of $vect$, which is grounded.
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First, we recall some known concepts and properties. A **hyperplane** of $E^n$ is a subset $H = \{ \vec{x} \in E^n | f(\vec{x}) = c \} = [f = c]$, where $f : E^n \to \mathbb{R}$ is a linear map. A hyperplane induces two open **halfspaces** $[f < c]$ and $[f > c]$ (open w.r.t. the Euclidean topology) and two closed ones $[f \leq c]$ and $[f \geq c]$. Every such halfspace is convex. We also know that any two different vectors $\vec{x}, \vec{y}$ are separated by a hyperplane, i.e. there is a linear map $f : E^n \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that $f(\vec{x}) < c$ and $f(\vec{y}) > c$.

We have the following properties.

**Proposition 1.3.2.** $(E^n,\text{vect})$ is not $T_0$, $(E^n,\text{aff})$ is $T_1$ but not Hausdorff and $(E^n,\text{conv})$ is Hausdorff.

**Proof.** It suffices to notice that any open set containing $\vec{o}$ in $(E^n,\text{vect})$ must be $E^n$. The $T_1$ property for $\text{aff}$ follows from $\text{aff}(\{\vec{s}\}) = \{\vec{s}\}$ and $\text{aff}$ is not Hausdorff since for any $A,B$ affine sets such that $A \cup B = E^n$, we have that one of them is $E^n$. $\text{conv}$ is Hausdorff because any two different vectors $\vec{x}, \vec{y}$ can be separated by a hyperplane. Since this hyperplane determines two disjoint halfspaces containing $\vec{x}$ resp. $\vec{y}$ and since their complements are convex, $\text{conv}$ is indeed a Hausdorff closure.

**Proposition 1.3.3.** $(E^n,\text{vect})$ and $(E^n,\text{aff})$ are connected closure spaces, $(E^n,\text{conv})$ is totally disconnected.

**Proof.** No proper open subset of $(E^n,\text{vect})$ contains $\vec{o}$, hence there are no proper clopen sets in $(E^n,\text{vect})$. For every proper affine subset $A$, $E^n \setminus A$ can not be affine, hence $(E^n,\text{aff})$ is connected. Suppose $A$ is a convex subset of $E^n$ containing $\vec{a}$. If there is another point $\vec{b} \in A$, then there is a hyperplane $[f = c]$ separating them, hence $[f < 0] \cap A$ and $[f \geq 0] \cap A$ are proper clopen sets in $(A,\text{conv})$. Therefore $K(\vec{a}) = \{\vec{a}\}$ and $(E^n,\text{conv})$ is totally disconnected.

From convex analysis we have the following proposition.

**Proposition 1.3.4.** Let $A,B$ be two disjoint convex subsets of $E^n$. Then there exist two convex sets $C$ and $D$ with $C \cap D = \emptyset$, $C \cup D = E^n$ such that $A \subset C$ and $B \subset D$.

The proof of this proposition uses Zorn’s lemma and can be found in many introductory textbooks about convex analysis, for instance in [46].
1. Closure spaces

**Proposition 1.3.5.** \((E^n, \text{conv})\) is a zero-dimensional closure space.

**Proof.** Let \(S\) be any convex set of \(E^n\) and consider \(S' = \bigcap \{D | S \subset D, D \text{ and } E^n \setminus D \text{ convex}\}\). Clearly \(S \subset S'\), in order to prove the converse we choose an \(\vec{x} \notin S\). Since \(\{\vec{x}\}\) and \(S\) are disjoint convex sets Proposition 1.3.4 yields a convex set \(D\) such that \(E^n \setminus D\) is also convex and \(S \subset D, \vec{x} \in E^n \setminus D\). Hence \(\vec{x} \notin S'\), so \(E^n \setminus S \subset E^n \setminus S'\) and therefore \(S' \subset S\). □

We conclude by summarizing the obtained results in the next table.

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1.3.2 In formal context Analysis

Formal context analysis is a theory aimed at giving a mathematical description of “concepts” and “concept hierarchy” within a certain context. It is widely used in the fields of data analysis and knowledge processing. Here we give a short introduction and show how the notion of closure space emerges. For more about this subject we refer to the introductory book [29] by B. Ganter and R. Wille.

**Definition 1.3.6.** A **context** is a triple \(\mathbb{K} = (G, M, I)\), where \(G\) is a set representing objects, \(M\) a set of attributes of these objects and \(I \subset G \times M\) a relation.

For an object \(g \in G\) and an attribute \(m \in M\) we will interpret \(gIm\) as “the object \(g\) has attribute \(m\)”. An easy way to represent a context is by means of a table.

\[
\begin{array}{c|c}
M & \\
\hline
G & \cdots \ x \\
\end{array}
\]
With a context $K = (G, M, I)$ one can associate two operations $\uparrow$ and $\downarrow$ as follows. For $A \subset G$, $A^\uparrow = \{ m \in M | \forall g \in A : gIm \}$ is the set of all attributes common to all objects of $A$. In the same way we define for $B \subset M$, $B^\downarrow = \{ g \in G | \forall m \in B : gIm \}$ to be the set of objects which posses all the attributes in $B$. With these operations one defines concepts of a context.

**Definition 1.3.7.** A concept of a context $K = (G, M, I)$ is a pair $(A, B)$ where $A \subset G$ and $B \subset M$, such that $A^\uparrow = B$ and $B^\downarrow = A$. The set of all concepts of a context will be written as $C_K$.

**Theorem 1.3.8.** [29] Let $K = (G, M, I)$ be a context. The operators $^\uparrow\downarrow$ and $^\downarrow\uparrow$ are closure operators on $G$ resp. $M$.

**Remark 1.3.9.** In general these closures are not grounded, however if we restrict ourselves to contexts were no object has all possible attributes and no attribute is possessed by all objects simultaneously, the closures $^\uparrow\downarrow$ and $^\downarrow\uparrow$ are grounded.

Furthermore if $(A, B)$ is a concept of a context $K = (G, M, I)$ one clearly sees that $A^\uparrow = B^\downarrow = A$, conversely for any set $A \subset G$, closed w.r.t. $^\uparrow\downarrow$, $(A, A^\uparrow)$ is a concept. Therefore one can endow $C_K$ with a partial order $(A_1, B_1) \leq (A_2, B_2) \iff A_1 \subset A_2$ (this is equivalent to $B_2 \subset B_1$) such that $C_K$ becomes a complete lattice, which is isomorphic to the complete lattice of closed sets of $(G, ^\uparrow\downarrow)$. From [29] it follows that this lattice is dually isomorphic with the complete lattice of closed sets of $(M, ^\downarrow\uparrow)$. This lattice is called the concept lattice of the context $K = (G, M, I)$.

We finish this section with an example.

![Concept lattice](image)
**Example 1.3.10.** Consider the context $\mathbb{K} = (G, M, I)$, with objects $G$ being the three closures form our first example $\{\text{vect}, \text{aff}, \text{conv}\}$ and attributes $M = \{T_0, T_1, T_2, \text{conn.}, \text{tot. disc.}, \text{zero-dim.}\}$. Let $I$ be the relation given by

<table>
<thead>
<tr>
<th>vect</th>
<th>$T_0$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>conn.</th>
<th>tot.disc.</th>
<th>zero-dim.</th>
</tr>
</thead>
<tbody>
<tr>
<td>aff</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>conv</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>

Then the concept lattice of $\mathbb{K}$ is given in Figure 1.3.

### 1.3.3 In quantum logic

In this section we will elaborate on the application of a certain equivalence between lattices and closure spaces in the context of the foundations of quantum mechanics. First we give a sketch of the reasoning that leads to the concept of state space and property lattice of a quantum entity $S$. Let us consider the Schrödinger equation:

$$H\psi = E\psi$$

describing the dynamics of the quantum entity $S$, where $H$ is a Hamiltonian and $E$ is the total energy of $S$. A particular state of $S$, i.e. a solution of the Schrödinger equation, is a vector $\psi$ in a Hilbert space $\mathcal{H}$.

In quantum mechanics the physical quantities of $S$, for instance impulse or position, are described by self adjoint operators on $\mathcal{H}$, the so-called “observables”. More specifically “yes/no” observables, i.e. observables with only two outcomes, are described by orthogonal projection operators on $\mathcal{H}$. An orthogonal projection is uniquely determined by the closed subspace of the Hilbert space that is its range, and that’s the reason that properties of $S$, corresponding to “yes/no” observables, can be described by the closed subspaces of the Hilbert space. At this point we note that these closed subspaces form a closure space, not a topological space. We can therefore reach the conclusion that the properties of $S$ form a complete lattice (of closed subspaces of $\mathcal{H}$), the **property lattice**. As said before this is just a sketch of how we get from the usual quantum mechanics (Schrödinger’s equation) to the logic of quantum mechanics (property lattices). For more details we refer the reader to the work of J. von Neumann and G. Birkhoff [16], [52], [15].
Using these ideas C. Piron started an axiomatic approach for the foundation of quantum mechanics [38] in Geneva. Together with D. Aerts in Brussels they elaborated this axiomatic operational approach to the foundations of quantum mechanics, which is now known as the Geneva-Brussels approach. Following this approach one starts by defining on a physical entity $S$ “tests”, which are experiments that can be performed on $S$ and either give rise to an outcome “yes” or an outcome “no”. By means of these “tests” a property lattice is constructed. A state is represented by the collections of all properties that are “actual” (the corresponding test gives with certainty the outcome “yes”) whenever the entity is in this state. We mention that in these early approaches [2], [3], [4], [38], [39] the mathematical structure that underlies the physical theory had not completely been identified. To identify the mathematical structure in a complete way, the structure of a state property system was introduced in [5] by D. Aerts.

Suppose that we consider a physical entity $S$, we denote its set of states by $\Sigma$ and its complete lattice of properties by $\mathcal{L}$, for which we will write $\top, \bot, \lor, \land$ for the maximal element, resp. the minimal element, suprema and infima. The state property system corresponding to this physical entity $S$ is a triple $(\Sigma, \mathcal{L}, \xi)$, where $\Sigma$ is the set of states of $S$, $\mathcal{L}$ the lattice of properties of $S$, and $\xi$ a map from $\Sigma$ to $\mathcal{P}(\mathcal{L})$, that makes correspond to each state $p \in \Sigma$ the set of properties $\xi(p) \in \mathcal{P}(\mathcal{L})$ that are actual if the entity $S$ is in state $p$. Some additional requirements, which express exactly how the physicists perceive a physical entity in relation with its states and properties, are satisfied in a state property system. Let us introduce the formal definition of a state property system and then explain what these additional requirements mean.

**Definition 1.3.11.** A triple $(\Sigma, \mathcal{L}, \xi)$ is called a state property system if $\Sigma$ is a set, $\mathcal{L}$ is a complete lattice and $\xi : \Sigma \rightarrow \mathcal{P}(\mathcal{L})$ is a function such that we have:

- (SP1) $\bot \not\in \xi(p)$
- (SP2) $\forall i \in I : a_i \in \xi(p) \Rightarrow \land_{i \in I} a_i \in \xi(p)$
- (SP3) $a \leq b \iff \forall r \in \Sigma : a \in \xi(r)$ implies $b \in \xi(r)$

where $p \in \Sigma$, $\bot$ is the minimal element of $\mathcal{L}$ and $(a_i)_{i \in I}, a, b \in \mathcal{L}$.

We demand that $\mathcal{L}$, the set of properties, is a complete lattice. This means that the set of properties is partially ordered, with the physical meaning of the
Closure spaces

partial order relation \( \leq \) being the following: \( a, b \in \mathcal{L} \), such that \( a \leq b \) means that whenever property \( a \) is actual for the entity \( S \), also property \( b \) is actual for the entity \( S \). If \( \mathcal{L} \) is a complete lattice, it means that for an arbitrary family of properties \( (a_i)_{i \in I} \) in \( \mathcal{L} \) also the infimum \( \wedge_{i \in I} a_i \) of this family is a property. The property \( \wedge_{i \in I} a_i \) is the property that is actual if and only if all of the properties \( a_i \) are actual. Hence the infimum represents the logical “and”. The minimal element \( \bot \) of the lattice of properties is the property that is never actual (e.g. the physical entity does not exist). Requirement (SP1) expresses that a property that is in the image by \( \xi \) of an arbitrary state \( p \in \Sigma \) can never be the \( \bot \) property. Requirement (SP2) expresses that if for a state \( p \in \Sigma \) all the properties \( a_i \) are actual, this implies that for this state \( p \) also the “and” property \( \wedge_{i \in I} a_i \) is actual. Requirement (SP3) expresses the meaning of the partial order relation that we gave already: \( a \leq b \) if and only if whenever \( p \) is a state of \( S \) such that \( a \) is actual if \( S \) is in this state, then also \( b \) is actual if \( S \) is in this state.

Along the same lines, just translating what the physicist means when he imagines the situation of two physical entities, of which one is a subentity of the other, the morphisms of state property systems can be deduced. More concretely, suppose that \( S \) is a subentity of \( S' \). Then each state \( p' \) of \( S' \) determines a state \( p \) of \( S \), namely the state \( p \) where the subentity \( S \) is in when \( S' \) is in state \( p' \). This defines a map \( m : \Sigma' \to \Sigma \). On the other hand, each property \( a \) of \( S \) determines a property \( a' \) of \( S' \), namely the property of the subentity, but now conceived as a property of the big entity. This defines a map \( n : \mathcal{L} \to \mathcal{L}' \).

Suppose that we consider now a state \( p' \) of \( S' \), and a property \( a \) of \( S \), such that \( a \in \xi(m(p')) \). This means that the property \( a \) is actual if the subentity \( S \) is in state \( m(p') \). This state of affairs can be expressed equally by stating that the property \( n(a) \) is actual when the big entity is in state \( p' \). Hence, as a basic physical requirement of covariance we should have:

\[
a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p')
\]

This all gives rise to the following definition of morphism for state property systems.

**Definition 1.3.12.** Suppose that \( (\Sigma, \mathcal{L}, \xi) \) and \( (\Sigma', \mathcal{L}', \xi') \) are state property systems then

\[
(m, n) : (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)
\]

is called an **SP-morphism** if \( m : \Sigma' \to \Sigma \) and \( n : \mathcal{L} \to \mathcal{L}' \) are functions such that for \( a \in \mathcal{L} \) and \( p' \in \Sigma' \):

\[
a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p')
\]
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Using the previous definitions we can obtain the category of state property systems.

**Definition 1.3.13.** The **category of state property systems** with their morphisms is denoted by $\mathbf{SP}$.

**Definition 1.3.14.** If $(\Sigma, \mathcal{L}, \xi)$ is a state property system then its **Cartan map** is the mapping $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$ defined by:

$$\kappa : \mathcal{L} \to \mathcal{P}(\Sigma) : a \mapsto \kappa(a) = \{p \in \Sigma | a \in \xi(p)\}$$

The state property systems defined as above constitute the mathematical framework for the Geneva-Brussels approach of the foundations of quantum mechanics. This physically inspired category $\mathbf{SP}$ of states property systems and its morphisms proved to be equivalent to the category $\mathbf{Cl}$ of closure spaces and the continuous maps.

The following proposition shows how we can associate with each state property system a closure space and with each morphism a continuous map, hence we get the categorical equivalence described in [6].

**Proposition 1.3.15.** The correspondence

$$F : \mathbf{SP} \to \mathbf{Cl} :$$

$$(\Sigma, \mathcal{L}, \xi) \mapsto F(\Sigma, \mathcal{L}, \xi) = (\Sigma, \kappa(\mathcal{L}))$$

$$(m, n) \mapsto m$$

where $\kappa(\mathcal{L})$ is considered as the collection of closed sets of $F(\Sigma, \mathcal{L}, \xi)$, is a functor.

We can also connect a state property system to a closure space and a morphism to a continuous map.

**Proposition 1.3.16.** The correspondence

$$G : \mathbf{Cl} \to \mathbf{SP} :$$

$$(\Sigma, \mathcal{C}) \mapsto G(\Sigma, \mathcal{C}) = (\Sigma, \mathcal{C}, \bar{\xi})$$

$$m \mapsto (m, m^{-1})$$

where $\bar{\xi} : x \mapsto \{C \in \mathcal{C} | x \in C\}$, is a functor.
1. Closure spaces

**Theorem 1.3.17.** The functors

\[ F : \text{SP} \rightarrow \text{Cl} \]

\[ G : \text{Cl} \rightarrow \text{SP} \]

establish an equivalence of categories.

For a detailed proof of the above theorem we refer to [6]. This equivalence is a very powerful tool for studying state property systems. It states that the lattice \( \mathcal{L} \) of properties can be seen as the lattice of closed sets of a closure space on the states \( \Sigma \), conversely every closure space on \( X \) can be considered as a set of states \( (X) \) and a lattice of properties (the lattice of closed sets). With the previous equivalence, a concept which can be defined using closed sets on a closure space can be translated in an equivalent concept for state property systems. At first sight this translation does not need to be meaningful in the context of physical systems. However it turns out that many such translations actually coincided with well-known physical concepts. We shall give one example which was studied in [47].

**Example 1.3.18.** [Atomisticity and \( T_1 \) separation.]

Let \( (\Sigma, \mathcal{L}, \xi) \) be a state property system. Then the map \( s_\xi \) maps a state \( p \) to the strongest property it makes actual, i.e.

\[ s_\xi : \Sigma \rightarrow \mathcal{L} : p \mapsto \wedge \xi(p) \]

The following are equivalent:

1. \( \xi : \Sigma \rightarrow \mathcal{P}(\mathcal{L}) \) is injective and \( \forall p \in \Sigma : s_\xi(p) \) is an atom of \( \mathcal{L} \).
2. \( \forall p, q \in \Sigma : \xi(p) \subset \xi(q) \Rightarrow p = q \)
3. \( F(\Sigma, \mathcal{L}, \xi) = (\Sigma, \kappa(\mathcal{L})) \) is a \( T_1 \) closure space.

If a state property system satisfies one, and hence all of the above conditions it is called an **atomistic state property system**, in this case \( \mathcal{L} \) is a complete atomistic lattice. If we write \( \text{SP}_a \) for the full subcategory of \( \text{SP} \) given by the atomistic state property systems, then the general equivalence can be reduced and functors

\[ F : \text{SP}_a \rightarrow \text{Cl}_1 \]

\[ G : \text{Cl}_1 \rightarrow \text{SP}_a \]

establish an equivalence of categories.
1. Closure spaces

For a more extensive study of separation axioms and their relation with state property systems we refer to [6], [51], [47]. In the next chapter we establish a link between zero-dimensionality and disconnectedness for closure spaces and the physical distinction between quantum mechanical and classical properties of a system.
Chapter 2

An application of closure spaces to quantum logic

In this chapter we will investigate in some more detail the relation introduced in Section 1.3.3 between closure spaces and state property systems. Our final aim is to use the equivalence described in Theorem 1.3.17 to translate the concept of connectedness for closure spaces into terms of state property systems. What is quite astonishing is that topological concepts as disconnectedness and zero-dimensionality will give us means to distinguish “classical” from “quantum mechanical” properties of a physical system. The results obtained in the following sections about the different sorts of classicality and the associated decomposition of state property systems were developed in collaboration with D. Aerts and A. De Groot-Van der Voorde. They were published in [9], [8] and in the book [7].

2.1 Superselection rules

In this section we start to distinguish the classical aspects of the structure from the quantum aspects. We know that the concept of superposition state is very important in quantum mechanics. The superposition states are the states that do not exist in classical physics and hence their appearance is one of the important quantum aspects. To be able to define properly a superposition state we need the linearity of the set of states. On the level of generality that we work now, we do not necessarily have this linearity, which could indicate that
the concept of superposition state cannot be given a meaning on this level of
generality. This is however not really true: the concept can be traced back
within this general setting, by introducing the idea of “superselection rule”.
Two properties are separated by a superselection rule if and only if there do not
exist “superposition states” related to these two properties. This concept will
be the first step towards a characterization of classical properties of a physical
system.

Definition 2.1.1. Consider a state property system \((\Sigma, \mathcal{L}, \xi)\). For \(a, b \in \mathcal{L}\) we
say that \(a\) and \(b\) are separated by a superselection rule, and denote \(a \text{ ssr } b\),
if and only if for \(p \in \Sigma\) we have:

\[
a \lor b \in \xi(p) \Rightarrow a \in \xi(p) \text{ or } b \in \xi(p)
\]

We again use the equivalence between state property systems and closure spaces
to translate the concept of “separation by a superselection rule” into a concept
for the closed sets of a closure space. Amazingly we find that properties that
are “separated by a superselection rule” (i.e. they are “classical” properties in
a certain sense) correspond to closed sets that also behave in a classical way,
where classical now refers to classical topology.

Proposition 2.1.2. Consider a state property system \((\Sigma, \mathcal{L}, \xi)\) and its corre-
sponding closure structure of closed sets \(\mathcal{C} = \kappa(\mathcal{L})\). For \(a, b \in \mathcal{L}\) we have:

\[
a \text{ ssr } b \iff \kappa(a \lor b) = \kappa(a) \cup \kappa(b) \iff \kappa(a) \cup \kappa(b) \in \mathcal{C}
\]

Proof. Suppose that \(a, b \in \mathcal{L}\) such that \(a \text{ ssr } b\). If \(p \in \kappa(a \lor b)\), then \(a \lor b \in \xi(p)\).
Then it follows \(a \in \xi(p)\) or \(b \in \xi(p)\). So we have \(p \in \kappa(a)\) or \(p \in \kappa(b)\), which
shows that \(p \in \kappa(a) \cup \kappa(b)\). This proves that \(\kappa(a \lor b) \subseteq \kappa(a) \cup \kappa(b)\). We
obviously have the other inclusion and hence \(\kappa(a \lor b) = \kappa(a) \cup \kappa(b)\). It follows
immediately that \(\kappa(a) \cup \kappa(b) \in \mathcal{C}\). Conversely, if \(\kappa(a) \cup \kappa(b) \in \mathcal{C}\), then there
exists a property \(c \in \mathcal{L}\) such that \(\kappa(c) = \kappa(a) \cup \kappa(b)\). From \(\kappa(a) \subseteq \kappa(c)\) it
follows that \(a \leq c\), and in a similar way we have \(b \leq c\). So \(a \lor b \leq c\), as a
consequence \(\kappa(a \lor b) \subseteq \kappa(c) = \kappa(a) \cup \kappa(b)\). Since \(\kappa(a) \cup \kappa(b) \subseteq \kappa(a \lor b)\), we have
\(\kappa(a \lor b) = \kappa(a) \cup \kappa(b)\). Consider now an arbitrary \(p \in \Sigma\) such that \(a \lor b \in \xi(p)\).
Then \(p \in \kappa(a \lor b) = \kappa(a) \cup \kappa(b)\). Hence \(p \in \kappa(a)\) or \(p \in \kappa(b)\). This proves that
\(a \in \xi(p)\) or \(b \in \xi(p)\) which shows that \(a \text{ ssr } b\).

This proposition shows that the properties that are separated by a superselection
rule are exactly the ones that behave also classically within the closure system.
2. An application of closure spaces to quantum logic

In the sense that their set theoretical unions are closed. This also means that if our closure system reduces to a topology, and hence all finite unions of closed subsets are closed, all finite sets of properties are separated by superselection rules.

**Corollary 2.1.3.** Let \((\Sigma, \mathcal{L}, \xi)\) be a state property system. The following are equivalent:

1. Every two properties of \(\mathcal{L}\) are separated by a superselection rule.
2. The corresponding closure space \((\Sigma, \kappa(\mathcal{L}))\) is a topological space.

A state property satisfying one, and hence both of the above conditions will be called a “superselection classical” state property system or **s-classical** state property system. The full subcategory of \(\text{SP}\) given by the \(s\)-classical state property systems will be written as \(\text{scSP}\).

Hence the equivalence between state property systems and closure space can be reduced to an equivalence between \(s\)-classical state property systems, in which no two properties have “superposition states” related to them, and topological spaces.

**Theorem 2.1.4.** The functors from Theorem 1.3.17 restrict as follows:

\[
F : \text{scSP} \rightarrow \text{Top}
\]

\[
G : \text{Top} \rightarrow \text{scSP}
\]

and they establish an equivalence of categories.

### 2.2 \(d\)-classical properties

Next we will introduce the concept of a “deterministic classical property” or “\(d\)-classical property”. To make clear what we mean by this we have to explain shortly how properties are tested in the earlier approaches mentioned in the introduction. For each property \(a \in \mathcal{L}\) there exists a test \(\alpha\), which is an experiment that can be performed on the physical entity under study, and that can give two outcomes, “yes” and “no”. The property \(a\) tested by the experiment \(\alpha\) is actual if and only if the state \(p\) of \(S\) is such that we can predict with certainty (probability equal to 1) that the outcome “yes” will occur for the test \(\alpha\). If the
state $p$ of $S$ is such that we can predict with certainty that the outcome “no” will occur, we test in some way a complementary property of the property $a$, let us denote the complementary property by $a^c$. Now we have three possibilities:

1. the state of $S$ is such that $\alpha$ gives “yes” with certainty;
2. the state of $S$ is such that $\alpha$ gives “no” with certainty;
3. the state of $S$ is such that neither the outcome “yes” nor the outcome “no” is certain for the experiment $\alpha$.

The third case represents the situations of “quantum indeterminism”. That is the reason that a property $a$ tested by an experiment $\alpha$ where the third case is absent will be called a “deterministic classical” property or “d-classical” property. Note that the notion of $d$-classicality is quite different from $s$-classicality: $s$-classical means classical in the sense that there do not exists superpositions (no “quantum superpositions”), however $d$-classical means that there is no uncertainty (no “quantum indeterminism”). We axiomize this as follows.

**Definition 2.2.1.** Consider a state property system $(\Sigma, \mathcal{L}, \xi)$. We say that a property $a \in \mathcal{L}$ is a “deterministic classical property” or $d$-classical property, if there exists a property $a^c \in \mathcal{L}$ such that $a \lor a^c = \top$, $a \land a^c = \bot$ and $a \text{ ssr } a^c$.

Remark that for every state property system $(\Sigma, \mathcal{L}, \xi)$ the properties $\bot$ and $\top$ are $d$-classical properties. Note also that if $a \in \mathcal{L}$ is a $d$-classical property, we have for $p \in \Sigma$ that $a \in \xi(p) \iff a^c \notin \xi(p)$. This follows immediately from the definition of a $d$-classical property.

**Proposition 2.2.2.** Consider a state property system $(\Sigma, \mathcal{L}, \xi)$. If $a \in \mathcal{L}$ is a $d$-classical property, then $a^c$ is unique and it is a $d$-classical property. We will call it the complement of $a$. Further we have, for $d$-classical properties $a, b \in \mathcal{L}$:

\[
\begin{align*}
(a^c)^c &= a \\
\mathcal{a} \leq b &\Rightarrow \mathcal{b}^c \leq a^c \\
\kappa(a^c) &= \Sigma \setminus \kappa(a)
\end{align*}
\]

**Proof.** Suppose that we have another property $b \in \mathcal{L}$ such that $a \lor b = \top$, $a \land b = \bot$ and $a \text{ ssr } b$. Consider an arbitrary state $p \in \Sigma$ such that $a^e \in \xi(p)$. This means that $a \notin \xi(p)$. We have however $a \lor b \in \xi(p)$, which implies, since
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...that we have proven that $a^c \leq b$. In a completely analogous way we can show that also $b \leq a^c$, which shows that $a^c$ is unique. Obviously $a^c$ is a $d$-classical property. Then the idempotency follows from the fact that $a$ is the complement of $a^c$ and from the uniqueness of the complement. Consider $a \leq b$ and an arbitrary state $p \in \Sigma$ such that $b^c \in \xi(p)$. This means that $b \notin \xi(p)$, which implies that $a \notin \xi(p)$. As a consequence we have $a \in \kappa(a)$.

Proposition 2.2.3. Consider a state property system $(\Sigma, L, \xi)$ and its corresponding closure space $(\Sigma, \kappa(L))$. For $a \in L$ we have:

$$a \text{ is } d\text{-classical } \iff \kappa(a) \text{ is clopen}$$

Proof. From the previous proposition it follows that if $a$ is $d$-classical, then $\kappa(a)$ is clopen. Let’s consider a clopen subset $\kappa(a)$ of $\Sigma$. This means that $\Sigma \setminus \kappa(a)$ is closed, and hence that there exists a property $b \in L$ such that $\kappa(b) = \Sigma \setminus \kappa(a)$. We clearly have $a \land b = \bot$ since there exists no state $p \in \Sigma$ such that $p \in \kappa(a)$ and $p \in \kappa(b)$. Since $\Sigma = \kappa(a) \cup \kappa(b)$ we have $a \lor b = \top$. Further we have that for an arbitrary state $p \in \Sigma$ we have $a \in \xi(p)$ or $b \in \xi(p)$ which shows that $a \ll sr b$. This proves that $b = a^c$ and that $a$ is $d$-classical.

Therefore the $d$-classical properties correspond exactly to the clopen subsets of the closure system.

Corollary 2.2.4. Let $(\Sigma, L, \xi)$ be a state property system. The following are equivalent:

1. The properties $\bot$ and $\top$ are the only $d$-classical ones.

2. $F(\Sigma, L, \xi) = (\Sigma, \kappa(L))$ is a connected closure space.

We now introduce “completely quantum mechanical” or pure nonclassical state property systems, in the sense that there are no (non-trivial) $d$-classical properties.

Definition 2.2.5. A state property system $(\Sigma, L, \xi)$ is called a pure nonclassical state property system if the properties $\bot$ and $\top$ are the only $d$-classical properties.
Proposition 2.2.6. Let $(\Sigma, \mathcal{C})$ be a closure space. The following are equivalent:

1. $(\Sigma, \mathcal{F})$ is a connected closure space.
2. $G(\Sigma, \mathcal{C}) = (\Sigma, \mathcal{C}, \bar{\xi})$ is a pure nonclassical state property system.

Proof. Let $(\Sigma, \mathcal{C})$ be a connected closure space. Then $\emptyset$ and $\Sigma$ are the only clopen sets in $(\Sigma, \mathcal{C})$. Since the Cartan map associated to $\bar{\xi}$ is given by $\kappa: \mathcal{C} \to \mathcal{P}(\Sigma): A \mapsto A$, we have $\kappa(\emptyset) = \emptyset$ and $\kappa(\Sigma) = \Sigma$. Applying Proposition 2.2.3, we find that $\emptyset$ and $\Sigma$ are the only $d$-classical properties of $\mathcal{C}$. Conversely, let $G(\Sigma, \mathcal{C}) = (\Sigma, \mathcal{C}, \bar{\xi})$ be a pure nonclassical state property system. Then by Corollary 2.2.4, $(\Sigma, \mathcal{C}) = FG(\Sigma, \mathcal{C})$ is a connected closure space.

If we define $\mathbf{SP}_Q$ as the full subcategory of $\mathbf{SP}$ where the objects are the pure nonclassical state property systems then the previous propositions and Theorem 1.3.17 imply an equivalence of the categories $\mathbf{SP}_Q$ and $\mathbf{ConnCl}$.

Theorem 2.2.7. Again the functors from Theorem 1.3.17 restrict:

$F: \mathbf{SP}_Q \to \mathbf{ConnCl}$

$G: \mathbf{ConnCl} \to \mathbf{SP}_Q$

and establish an equivalence of categories.

Again we have found using the equivalence 1.3.17 that a physical concept (i.e. nonclassicality) translates to a known topological property (i.e. connectedness). Using the correspondence from Proposition 2.2.3 between clopen subset and classical properties we can make yet another equivalence.

Definition 2.2.8. An atomistic state property system $(\Sigma, \mathcal{L}, \xi)$ is called a totally classical state property system if the only pure nonclassical segments of $\mathcal{L}$ (i.e. segments with no proper $d$-classical elements) are trivial, i.e. $\{\bot, a\}$ where $a$ is an atom.

If we write the full subcategory of $\mathbf{SP}_a$ generated by the totally classical state property systems as $^t\mathbf{SP}_a$ the equivalence yields the following theorem.

Theorem 2.2.9. The functors

$F: ^t\mathbf{SP}_a \to \mathbf{TDiscCl}$

$G: \mathbf{TDiscCl} \to ^t\mathbf{SP}_a$

establish an equivalence of categories.
We do need the fact that \((\Sigma, \mathcal{L}, \xi)\) is atomistic as is shown by the following example.

**Example 2.2.10.** [A state property system which is totally classical but not atomistic, such that the associated closure is not totally disconnected.]

Let \(\Sigma = \{p, q, r\}\) and choose \(\mathcal{L}\) as in Figure 2.1 and \(\xi\) given by

\[
\begin{align*}
\xi : \Sigma &\to \mathcal{P}(\mathcal{L}) \\
p &\mapsto \{2, 4\} \\
q &\mapsto \{2, 4\} \\
r &\mapsto \{3, 4\}
\end{align*}
\]

Then \((\Sigma, \mathcal{L}, \xi)\) is not atomistic, totally classical and the corresponding closure space is \((\Sigma, \{\emptyset, \{p, q\}, \{r\}, \Sigma\})\), which is clearly not totally disconnected since it is not \(T_1\).

![Figure 2.1: The lattice \(\mathcal{L}\)](image)

In the next section we will use topological methods to construct a decomposition of a state property system into pure nonclassical components, analogous to the decomposition of a closure space into connected subspaces.

### 2.3 Decomposition theorem

As for topological spaces, every closure space can be decomposed uniquely into connected components (see Proposition 1.2.18). In the following we will try to decompose state property systems similarly into different components.

**Proposition 2.3.1.** Let \((\Sigma, \mathcal{L}, \xi)\) be a state property system and let \((\Sigma, \kappa(\mathcal{L}))\) be the corresponding closure space. Consider the following equivalence relation on \(\Sigma\), induced by the connection relation of the closure space:

\[ p \sim q \iff pKq \]
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with equivalence classes \( \Omega = \{ \omega(p) | p \in \Sigma \} \). If \( \omega \in \Omega \) we define the following:

\[
\Sigma_\omega = \omega = \{ p \in \Sigma | \omega(p) = \omega \}
\]

\[
s(\omega) = s(\omega(p)) = a, \text{ such that } \kappa(a) = \omega(p)
\]

\[
L_\omega = [\bot, s(\omega)] = \{ a \in L | \bot \leq a \leq s(\omega) \} \subset L
\]

\[
\xi_\omega : \Sigma_\omega \rightarrow \mathcal{P}(L_\omega) : p \mapsto \xi(p) \cap L_\omega
\]

then \((\Sigma_\omega, L_\omega, \xi_\omega)\) is a state property system.

\textbf{Proof.}\ Since \( L_\omega \) is a sublattice (segment) of \( L \), it is a complete lattice with maximal element \( \top_\omega = s(\omega) \) and minimal element \( \bot_\omega = \bot \). Let \( p \in \Sigma_\omega \). Then \( \bot \notin \xi(p) \). So \( \bot \notin \xi(p) \cap L_\omega = \xi_\omega(p) \). If \( a_i \in \xi_\omega(p), \forall i \in I \), then \( a_i \in L_\omega \) and \( a_i \in \xi(p), \forall i \in I \). Hence \( \bigwedge_{i \in I} a_i \in L_\omega \cap \xi(p) = \xi_\omega(p) \). Finally, let \( a, b \in L_\omega \) with \( a \leq_\omega b \) and let \( r \in \Sigma_\omega \). If \( a \in \xi_\omega(r) \), then \( a \in L_\omega \) and \( a \in \xi(r) \), thus \( b \in L_\omega \) and \( b \in \xi(r) \). Conversely, if \( a, b \in L_\omega \) and \( \forall r \in \Sigma_\omega : a \in \xi_\omega(r) \Rightarrow b \in \xi_\omega(r) \) then we consider a \( q \) such that \( a \in \xi(q) \) (\( q \) must be in \( \Sigma_\omega \) by definition of \( L_\omega \)). Then \( a \in \xi_\omega(q) \) implies that \( b \in \xi_\omega(q) \). So \( b \in \xi(q) \) and \( a \leq b \). Thus \( a \leq_\omega b \).

Moreover the state property systems \((\Sigma_\omega, L_\omega, \xi_\omega), \omega \in \Omega \) introduced above have no proper d-classical properties, and hence are pure nonclassical state property systems.

\textbf{Proposition 2.3.2.}\ Let \((\Sigma, L, \xi)\) be a state property system. If \( \omega \in \Omega \), then \((\Sigma_\omega, L_\omega, \xi_\omega)\) is a pure nonclassical state property system.

\textbf{Proof.}\ If \( a \) is classical element of \( L_\omega \), then \( \kappa(a) \) must be a clopen set of the associated closure space \((\Sigma_\omega, \kappa(L_\omega))\) which is a connected subspace of \((\Sigma, \kappa(L))\). Hence there are no proper classical elements of \( L_\omega \).

\textbf{Proposition 2.3.3.}\ Let \((\Sigma, L, \xi)\) be a state property system. If we introduce the following:

\[
\Omega = \{ \omega(p) | p \in \Sigma \}
\]

\[
K = \{ \vee_{i \in I} s(\omega_i) | i \in I, \omega_i \in \Omega \}
\]

\[
\eta : \Omega \rightarrow \mathcal{P}(K) : \omega = \omega(p) \mapsto \xi(p) \cap K
\]

then \((\Omega, K, \eta)\) is an atomistic state property system.
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\[ \text{Proposition 2.3.4. } (\Omega, \mathcal{K}, \eta) \text{ is a totally classical state property system.} \]

Proof. Suppose \([\bot, a]\) is a pure nonclassical segment of \(\mathcal{K}\), then in the corresponding closure space \((\Sigma, \kappa(\mathcal{L}))\) the subset \(\kappa(a)\) is connected hence \(a \leq \omega\) for some \(\omega \in \Omega\), hence \(a \leq s(\omega)\). Since \(s(\omega)\) is an atom, \(a = s(\omega)\). Thus \([\bot, a] = \{\bot, s(\omega)\}\). \(\blacksquare\)

\[ \text{Corollary 2.3.5. } \text{The closure space associated with } (\Omega, \mathcal{K}, \eta) \text{ is a totally disconnected closure space, in fact it is the } \text{TDiscCl} \text{ extremal epireflection of } (\Sigma, \kappa(\mathcal{L})). \]

Summarizing the previous results we get the following decomposition theorem:

\[ \text{Theorem 2.3.6. } \text{Any state property system } (\Sigma, \mathcal{L}, \xi) \text{ can be decomposed into:} \]

- a number of pure nonclassical state property systems \((\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)\), \(\omega \in \Omega\)
- a totally classical state property system \((\Omega, \mathcal{K}, \eta)\)

Thus the decomposition of a closure space into its connection components yields a way to decompose a state property system \((\Sigma, \mathcal{L}, \xi)\) into pure nonclassical state property systems \((\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)\), \(\omega \in \Omega\). In the context of closure spaces the
maximal connected components are subspaces of the given space. However we
do not yet have that the pure nonclassical state property systems \((\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)\)
are subsystems of \((\Sigma, \mathcal{L}, \xi)\). To show this we introduce a suitable concept of
subsystem.

2.4 \textit{ap}-subsystems

Definition 2.4.1. Let \((\Sigma, \mathcal{L}, \xi)\) be a state property system and let \(a \in \mathcal{L}\). Con-
sider the following:

\[
\Sigma' = \kappa(a)
\]
\[
\mathcal{L}' = [\bot, a]
\]
\[
\xi' = \xi_{\Sigma'}
\]

We now have a new state property system \((\Sigma', \mathcal{L}', \xi')\) which we shall call an
“actual property”-subsystem or, more briefly, an \textbf{ap-subsystem} of \((\Sigma, \mathcal{L}, \xi)\)
generated by \(a\).

The name “actual property”-subsystem comes from the physical interpretation
of this construction: give a property \(a\) of the physical system, we consider
only those states \(\Sigma'\) for which \(a\) is always actual. The next propositions follow
immediately.

Proposition 2.4.2. Let \((\Sigma', \mathcal{L}', \xi')\) be an \textbf{ap-subsystem} of \((\Sigma, \mathcal{L}, \xi)\), generated
by \(a\). Consider the corresponding closure spaces \((\Sigma', \kappa(\mathcal{L}'))\) and \((\Sigma, \kappa(\mathcal{L}))\), we
have that \((\Sigma', \kappa(\mathcal{L}'))\) is a closed subspace of \((\Sigma, \kappa(\mathcal{L}))\).

Proposition 2.4.3. Consider a closed subspace \((\Sigma', \mathcal{C}_{\Sigma'})\) of the closure space
\((\Sigma, \mathcal{C})\), we have that \((\Sigma', \mathcal{C}_{\Sigma'}, \xi')\) is an \textbf{ap-subsystem} of \((\Sigma, \mathcal{C}, \xi)\) generated by \(\Sigma'\).

Hence \textbf{ap-subsystems} correspond exactly to closed subspaces of the associated
closure space.

Any closed subspace \(\Sigma'\) of a closure space \((\Sigma, \mathcal{C})\) induces in a natural way a
canonical inclusion map:

\[
i : (\Sigma', \mathcal{C}_{\Sigma'}) \rightarrow (\Sigma, \mathcal{C})
\]

which in turn, by the functional equivalence between the category of closure
spaces and state property systems yields a \textbf{SP}-morphism:

\[
(i, i^{-1}) : (\Sigma', \mathcal{C}_{\Sigma'}, \xi') \rightarrow (\Sigma, \mathcal{C}, \xi)
\]
Theorem 2.4.4. Let $(\Sigma', L', \xi')$ be an ap-subsystem of $(\Sigma, L, \xi)$, generated by $a$. We now define the following maps:

$$m : \Sigma' \to \Sigma : p \mapsto p$$

$$n : L \to L' : c \mapsto a \land c$$

then $(m, n) : (\Sigma', L', \xi') \to (\Sigma, L, \xi)$ is a morphism in the category of state property systems which reduces to the canonical inclusion between the underlying closure spaces.

Proof. We have to show that for $c \in L$ and $p' \in \Sigma'$: $c \in \xi(m(p')) \iff n(c) \in \xi'(p')$. Let’s start with $c \in \xi(m(p')) \iff c \in \xi(p') \iff c \in \xi'(p')$. Because $\kappa(a) = \Sigma'$ we know that $a \in \xi'(p') = \xi(p')$, therefore $n(c) = c \land a \in \xi'(p')$. Conversely, if $n(c) = c \land a \in \xi'(p')$ then $p' \in \kappa'(c \land a) = \kappa'(c) \cap \kappa'(a) = \kappa'(c) \cap \Sigma' = \kappa'(c)$ therefore $c \in \xi'(p')$.

We shall apply these results to the pure nonclassical state property systems $(\Sigma_\omega, L_\omega, \xi_\omega), \omega \in \Omega$ that we have introduced in the previous section. Recall that we started with a state property system $(\Sigma, L, \xi)$ with associated closure space $(\Sigma, \kappa(L))$. By means of the connection relation $K$ on $(\Sigma, \kappa(L))$ we obtained a partition $\Omega = \{\omega(p) = K(p) | p \in \Sigma\}$ of $\Sigma$. Moreover each $w \in \Omega$ with $\omega = \omega(p) = K(p)$ was a closed subset of $(\Sigma, \kappa(L))$. Hence there was an $a = s(\omega)$ such that $\kappa(a) = \omega$. We will now use this property $a = s(\omega)$ to create an ap-subsystem.

$$\Sigma' = \kappa(a) = \omega$$

$$L' = [\bot, a] = [\bot, s(\omega)]$$

$$\xi' = \{ \xi | \Sigma' : p' \mapsto \xi(p) \cap L' \}$$

We easily see that for an $\omega \in \Omega$ this ap-subsystem is in fact $(\Sigma_\omega, L_\omega, \xi_\omega)$. Let

$$m : \Sigma_\omega \to \Sigma : p \mapsto p$$

$$n : L \to L_\omega : c \mapsto s(\omega) \land c$$

then $(m, n) : (\Sigma', L', \xi') \to (\Sigma, L, \xi)$ is a morphism in the category of state property systems which reduces to the canonical inclusion between the underlying closure spaces. In this way $(\Sigma_\omega, L_\omega, \xi_\omega), \omega \in \Omega$ is always an ap-subsystem of $(\Sigma, L, \xi)$.
2.5 The $d$-classical part of a state property system

In this section we want to show how it is possible to extract the $d$-classical part of a state property system. First of all we have to define the $d$-classical property lattice related to the entity $S$ that is described by the state property system $(\Sigma, L, \xi)$.

**Definition 2.5.1.** Consider a state property system $(\Sigma, L, \xi)$. We call $L^{dc} = \{ \land_{i \in I} a_i \mid (a_i)_{i \in I} \text{ are } d\text{-classical properties} \}$ the $d$-classical property lattice corresponding to the state property system $(\Sigma, L, \xi)$. $\blacktriangle$

**Proposition 2.5.2.** $L^{dc}$ is a complete lattice with the partial order relation and infimum inherited from $L$ and the supremum defined as follows: for elements $(a_i)_{i \in I}$ of $L^{dc}$, $\lor_{i \in I} a_i = \land_{b \in L^{dc}, a_i \leq b_{i \in I}} b$.

Note that the supremum in the lattice $L^{dc}$ is not the one inherited from $L$.

**Proposition 2.5.3.** Consider a state property system $(\Sigma, L, \xi)$. Let $\xi^{dc}(q) = \xi(q) \cap L^{dc}$ for $q \in \Sigma$, then $(\Sigma, L^{dc}, \xi^{dc})$ is a state property system which we shall refer to as the $d$-classical part of $(\Sigma, L, \xi)$.

**Proof.** Clearly $\bot \notin \xi^{dc}(p)$ for $p \in \Sigma$. Consider $a_i \in \xi^{dc}(p) \forall i \in I$. Then $a_i \in \xi(p) \cap L^{dc} \forall i \in I$, from which follows that $\land_{i \in I} a_i \in \xi(p) \cap L^{dc}$ and hence $\land_{i \in I} a_i \in \xi^{dc}(p)$. Consider $a, b \in L^{dc}$. Let us suppose that $a \leq b$ and consider $r \in \Sigma$ such that $a \in \xi^{dc}(r)$. This means that $a \in \xi(r) \cap L^{dc}$. From this follows that $b \in \xi(r) \cap L^{dc}$ and hence $b \in \xi^{dc}(r)$. On the other hand let us suppose that $\forall r \in \Sigma : a \in \xi^{dc}(r)$ then $b \in \xi^{dc}(r)$. Since $a, b \in L^{dc}$, this also means that $\forall r \in \Sigma : a \in \xi(r)$ then $b \in \xi(r)$. From this follows that $a \leq b$.

Since $(\Sigma, L^{dc}, \xi^{dc})$ is a state property system, it has a corresponding closure space $(\Sigma, \kappa(L^{dc}))$.

**Theorem 2.5.4.** The closure space $(\Sigma, \kappa(L^{dc}))$ corresponding to the state property system $(\Sigma, L^{dc}, \xi^{dc})$ is weakly zero-dimensional, in fact it is the $0\text{Cl}$-bi-reflection of $(\Sigma, \kappa(L))$.

**Proof.** To see this recall that $a$ is classical if and only if $\kappa(a)$ is clopen in $(\Sigma, \kappa(L))$, hence $\kappa(L^{dc})$ is a family of closed sets on $\Sigma$ which consists of all intersections of the clopen sets of $(\Sigma, \kappa(L))$. $\blacksquare$
In general \((\Sigma, \mathcal{L}^{dc}, \xi^{dc})\) does not need to be atomistic, hence it is different from the totally classical state property system \((\Omega, \mathcal{K}, \eta)\) associated with \((\Sigma, \mathcal{L}, \xi)\). To illustrate this we give an example.

**Example 2.5.5.** [A state property system \((\Sigma, \mathcal{L}, \xi)\) such that \((\Sigma, \mathcal{L}^{dc}, \xi^{dc})\) and \((\Omega, \mathcal{K}, \eta)\) are different.]

Let’s consider the following state property system.

\[
\begin{align*}
\Sigma &= \{p, q, r, s, t\} \\
\mathcal{L} &= \{1, 2, 3, 4, 5, 6\} \\
\xi : \Sigma &\to \mathcal{P}(\mathcal{L})
\end{align*}
\]

with \(\xi(p) = \xi(q) = \{3, 5, 6\}, \xi(r) = \{4, 5, 6\}\) and \(\xi(s) = \xi(t) = \{2, 6\}\). The structure for the lattice \(\mathcal{L}\) is given by Figure 2.2.

\[\text{Figure 2.2: The lattice } \mathcal{L}\]

The corresponding closure space (see Figure 2.3) is

\[
\begin{align*}
\Sigma &= \{p, q, r, s, t\} \\
\kappa(\mathcal{L}) &= \{\emptyset, \{r\}, \{p, q\}, \{s, t\}, \{p, q, r\}, \Sigma\}
\end{align*}
\]

Determining the connectedness components in this closure space, we find the following:

\[
\begin{align*}
K(p) &= K(q) = \{p, q\} \\
K(r) &= \{r\} \\
K(s) &= K(t) = \{s, t\}
\end{align*}
\]

We have three pure nonclassical state property systems: \((\Sigma_{\omega_1}, \mathcal{L}_{\omega_1}, \xi_{\omega_1})\), \((\Sigma_{\omega_2}, \mathcal{L}_{\omega_2}, \xi_{\omega_2})\) and \((\Sigma_{\omega_3}, \mathcal{L}_{\omega_3}, \xi_{\omega_3})\).

\[
\begin{align*}
\Sigma_{\omega_1} &= \{p, q\}, \quad \mathcal{L}_{\omega_1} = [\bot, 3] \\
\Sigma_{\omega_2} &= \{r\}, \quad \mathcal{L}_{\omega_2} = [\bot, 4] \\
\Sigma_{\omega_3} &= \{s, t\}, \quad \mathcal{L}_{\omega_3} = [\bot, 2]
\end{align*}
\]
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Figure 2.3: The closure space $\Sigma, \kappa(\mathcal{L})$

$$\xi_{\omega_1}(p) = \xi_{\omega_1}(q) = \{3\}$$  
$$\xi_{\omega_2}(r) = \{4\}$$  
$$\xi_{\omega_3}(s) = \xi_{\omega_3}(t) = \{2\}$$

The atomistic totally classical state property system $(\Omega, \mathcal{K}, \eta)$ is given by:

$$\Omega = \{\{p, q\}, \{r\}, \{s, t\}\}$$  
$$\mathcal{K} = \mathcal{L}$$  
$$\eta : \Omega \to \mathcal{P}(\mathcal{K})$$

where $\eta(\{p, q\}) = \{3, 5, 6\}$, $\eta(\{r\}) = \{4, 5, 6\}$ and $\eta(\{s, t\}) = \{2, 6\}$. The classical part is given by $(\Sigma, \mathcal{L}^{dc}, \xi^{dc})$ where

$$\xi^{dc}(p) = \xi(p) \cap \mathcal{L}^{dc} \text{ for } p \in \Sigma$$

$$\mathcal{L}^{dc} = \{1, 2, 5, 6\}$$
Chapter 3

Representations of closure spaces

Often in topology a space is represented by its lattice of open subsets or the lattice of its closed subsets. As for topological spaces lattices arise naturally in the closure context, since for any closure space $X$ both the collections of open subsets $\mathcal{O}_X$ and closed subsets $\mathcal{C}_X$, ordered by inclusion, form complete lattices.

For instance, choosing the open sets in order to generate a lattice gives in the topological setting the concepts of frame and locale. We will first turn our attention to developing a theory similar to the theory of spatial frames and sober topological spaces. We will follow the approach of lattices of open sets as is done in the topological case (P.T. Johnstone [33], B. Banaschewski [14]). This will give us a dual equivalence between the category of complete lattices with $\lor, \top$-preserving maps and a certain subcategory of $\mathbf{Cl}_0$ consisting of “point-closure” spaces.

Afterwards we will closely follow the work of M. Erné [26] and C.L. Faure [27], using lattices of closed sets. This will allow us to describe some concrete equivalences between certain categories of $T_0$ closure spaces and categories of complete lattices. We will find the previously described duality as a special case.

Finally, in the classical setting there is the well-known Stone representation theorem, which describes a dual equivalence between the category of Boolean algebra's and the category of compact zero-dimensional Hausdorff topological spaces. This representation is based on the observation that the set of clopen
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subsets of a topological space forms not only a complete lattice but a Boolean algebra. We will give a similar representation theorem for closure spaces, however we will consider complemented posets instead of Boolean algebras since for a closure space the set of clopen subsets forms only a partial order and not a complete lattice. The representation theorem we formulate, has already been studied in the context of quantum logic and orthocomplemented posets [55],[35]. We will place it in our context of zero-dimensional closure spaces and introduce an appropriate concept of compactness, using convergence of certain stacks.

3.1 A Duality

In this section we will construct a duality similar to the one between spatial frames and sober topological spaces. We will follow a path, intended to keep as close as possible to the topological situation. This will allow us to point out several differences between the topological and the closure context, it will also prepare us to what will come when we look at completeness later on.

We recall that, for a complete lattice $L$, we will write the top element as $\top$ and the bottom one as $\bot$, suprema (resp. infima) will be written as $\lor$ (resp. $\land$). For more about lattice theory we refer to [15] and for the categorical concepts as functors and equivalent categories we again use [1] as a reference.

For the general results in the theory of frames we refer the reader to [33] and [14]. As in frame theory we start with the observation that the collection $\mathcal{O}_X$ of open subsets of a closure space $X$ forms a complete lattice and that for any continuous function $f: X \to Y$ the map $f^{-1}: \mathcal{O}_Y \to \mathcal{O}_X$ preserves arbitrary unions and $f^{-1}(Y) = X$, so it also preserves the top element $\top$. This leads to the category of complete lattices with $\lor$, $\top$-preserving maps $\mathbf{CLat}_{\lor, \top}$. The dual or opposite category will be denoted $\mathbf{CLat}_{\lor, \top}^{op}$.

**Proposition 3.1.1.** We have the following functor:

\[
\begin{align*}
O: \mathbf{Cl} & \to \mathbf{CLat}_{\lor, \top}^{op} \\
X & \mapsto \mathcal{O}_X \\
f & \mapsto O(f)
\end{align*}
\]

where for $f: X \to Y$ we have that $O(f)$ is given by the $\mathbf{CLat}_{\lor, \top}$-morphism $f^{-1}: \mathcal{O}_Y \to \mathcal{O}_X: B \mapsto f^{-1}(B)$. 

\[47\]
Our aim is to find a suitable adjoint functor $S : \text{CLat}_{\lor,\top} \to \text{Cl}$. Inspired by frame theory we notice that a point $x \in X$ in a closure space $X$ can be seen as a morphism by means of the inclusion $i : (\{x\}, \{\emptyset, \{x\}\}) \to X$. This morphism is mapped by $O$ to a surjective $\text{CLat}_{\lor,\top}$-morphism $i^{-1} : O_X \to 2$, where $\mathcal{2} = \{\bot, \top\}$ is the two point complete lattice. Thus we generalize the concept of a point.

**Definition 3.1.2.** A point in a complete lattice $L$ is a surjective $\text{CLat}_{\lor,\top}$-morphism $\xi : L \to 2$. In the sequel we’ll use $\text{pts}(L)$ to denote the set of points of $L$ and for $a \in L$ we’ll write $\Sigma_a = \{\xi \in \text{pts}(L) | \xi(a) = \top\}$. ◀

**Remark 3.1.3.** In contrast to the topological and frame counterpart, for $a$ and $b$ in $L$, we always have that $a \neq b$ implies $\Sigma_a \neq \Sigma_b$.

**Proof.** Let $L$ be a complete lattice and $a \neq b \in L$. Suppose $a \nleq b$, then $\xi : L \to 2$ defined by $\xi(c) = \bot \iff c \leq b$ is a point such that $\xi(b) = \bot$ and $\xi(a) = \top$. If $a \leq b$ then $\xi' : L \to 2$ defined by $\xi'(c) = \bot \iff c \leq a$ is a point such that $\xi'(a) = \bot$ and $\xi'(b) = \top$. Hence $\Sigma_a \neq \Sigma_b$. ◀

Hence in the context of closure spaces all complete lattices are “spatial”. ◀

**Proposition 3.1.4.** We have the following functor:

$$S : \text{CLat}_{\lor,\top}^{\text{op}} \to \text{Cl} :$$

$$L \mapsto S(L)$$

$$\phi \mapsto S(\phi)$$

where $S(L)$ denotes the closure space on $\text{pts}(L)$ with open sets $\{\Sigma_a | a \in L\}$ and for a $\text{CLat}_{\lor,\top}$-morphism $\phi : L \to L'$, $S(\phi)$ is the continuous function $S(\phi) : S(L') \to S(L) : \xi' \mapsto \xi' \circ \phi$.

**Proof.** Since every point $\xi : L \to 2$ is $\lor$-preserving we have $\cup_{i \in I} \Sigma_{a_i} = \Sigma_{\vee_{i \in I} a_i}$ and $\Sigma_\top = \text{pts}(L)$, hence $S(L)$ is a closure space. If $\phi : L \to L'$ and $\xi' : L' \to 2$ is a point of $L'$ then $\xi' \circ \phi : L \to 2$ is a point of $L$. Functoriality follows from $S(\phi \circ \psi)(\xi) = \xi \circ \phi \circ \psi = S(\psi)(\xi \circ \phi) = S(\psi) \circ S(\phi)(\xi)$, where $\psi : L \to L'$, $\phi : L' \to L''$ and $\xi \in \text{pts}(L'')$. ◀

In order to obtain the desired duality, we’ll need the concept of a “complete” $T_0$ closure space. We first introduce some special stacks.
3. Representations of closure spaces

**Definition 3.1.5.** Let $X$ be a closure space. A proper open based stack $A$ is said to be a fundamental stack if it contains a member of every open cover of every element of $A$, i.e., $A \in A$ and $G$ an open cover of $A$ then $\exists G \in G : G \in A$. Shortly a proper fundamental open based stack is called an $O$-stack.

Note that in every closure space $X$ the neighborhood collection $V_X(x)$ of a point $x$ is an $O$-stack.

**Proposition 3.1.6.** On a closure space $X$ and for $A \subset \mathcal{P}(X)$ we have: $A$ is an $O$-stack if and only if there exists a (closed) nonempty set $F \subset X$ such that $A = \text{stack} \{G \subset X | G \text{ open}, G \cap F \neq \emptyset\}$.

**Proof.** For $F$ nonempty it is clear that

$$A = \text{stack} \{G | G \text{ open}, G \cap F \neq \emptyset\} = \text{stack} \{G | G \text{ open}, G \cap \text{cl}_X F \neq \emptyset\}$$

is an $O$-stack.

Conversely let $A$ be an $O$-stack. Let $F = \{x \in X | V_X(x) \subset A\}$. $F$ clearly is nonempty since otherwise there would exist an open cover of $X$ of which all members are not in $A$. If $G$ is open and $G \cap F \neq \emptyset$ then $V_X(x) \subset A$ for some point $x \in G$. So $G \in A$. On the other hand, if $G$ is open and belongs to $A$ then $G$ has to intersect $F$. If not, there would exist an open cover of $G$ of which all members are not in $A$. So finally we can conclude that

$$A = \text{stack} \{G | G \text{ open}, G \cap F \neq \emptyset\}$$

Remark that the set $F = \{x \in X | V_X(x) \subset A\}$ is in fact closed. □

**Definition 3.1.7.** A $T_0$ closure space $X$ is called a complete $T_0$ closure space if every $O$-stack is a neighborhood collection $V_X(x)$ for some (unique) point $x \in X$. The full subcategory of $\text{Cl}_0$ given by the complete $T_0$ closure spaces will be written as $\text{CCl}_0$.

The uniqueness of the point follows from the $T_0$ condition. Later on we will see that these spaces behave as one expects from complete objects in a more general categorical setting.

In view of Proposition 3.1.6 we get the following equivalent description.
3. Representations of closure spaces

**Theorem 3.1.8.** A $T_0$ closure space $X$ is complete if and only if every nonempty closed set is the closure of a (unique) point.

**Proof.** If $F$ is closed and nonempty then there is a point $x \in X$ such that stack $\{G | G \text{ open, } G \cap F \neq \emptyset\} = V_X(x)$. Then clearly $F = \text{cl}_X \{x\}$.

Conversely if $A$ is an $O$-stack, as in the proof of Proposition 3.1.6, let $F$ be the nonempty closed set $F = \{x \in X | V_X(x) \subset A\}$.

Now $F = \text{cl}_X \{x\}$ implies $A = V_X(x)$. □

The previous theorem justifies the fact that the complete $T_0$ closure spaces are also called point-closure spaces [26]. They also appear in [22] where Y. Diers calls them algebraic closure spaces.

**Proposition 3.1.9.** For a complete lattice $L$, $S(L)$ is a complete $T_0$ closure space.

**Proof.** Consider two distinct $\xi_1, \xi_2 \in \text{pts}(L)$ of $S(L)$. There exist a $a \in L$ such that $\xi_1(a) \neq \xi_2(a)$. Hence either $\xi_1 \in \Sigma_a$ and $\xi_2 \notin \Sigma_a$ or $\xi_1 \notin \Sigma_a$ and $\xi_2 \in \Sigma_a$. So $S$ is $T_0$. To prove completeness we choose an $O$-stack $A$ in $S(L)$ and consider $b = \vee\{a \in L | \Sigma_a \notin A\}$. Next we define the point $\xi : L \rightarrow 2 : a \mapsto \begin{cases} \top & a \notin b \\ \bot & b \leq a \end{cases}$. We have that $\xi(b) = \bot$, hence $(\Sigma_a \notin A \Rightarrow \xi(a) = \bot)$. Conversely, if $\Sigma_a \in A$ then $a \notin b$ since $A$ is an $O$-stack. Thus $(\Sigma_a \in A \Rightarrow \xi(a) = \top)$. Finally $\Sigma_a \in A \iff \xi(a) = \top$ and therefore $A = V_{S(L)}(\xi)$. □

**Theorem 3.1.10.** The restrictions

$O : \text{CCl}_0 \rightarrow \text{CLat}_{\vee, \top}^{\text{op}}$

$S : \text{CLat}_{\vee, \top}^{\text{op}} \rightarrow \text{CCl}_0$

define an equivalence of categories.

**Proof.** The proof consists of three parts.

1. Let $L$ be a complete lattice then $L \simeq OS(L)$. Choose the isomorphism as follows:

$\epsilon_L : OS(L) \rightarrow L : \Sigma_a \mapsto a$

This is a well-defined $\text{CLat}_{\vee, \top}^{\text{op}}$-isomorphism.
2. Let \( X \) be a complete \( T_0 \) closure space then \( X \simeq SO(X) \). Define the following map

\[
\eta_X : X \to pts(O(X)) : x \mapsto \xi_x
\]

where \( \xi_x(A) = \{ \top \mid x \in A \} \cup \{ \bot \mid x \notin A \} \), for all open sets \( A \).

\( \eta_X \) is injective since by the \( T_0 \) property we have for \( x \neq y \) an open subset \( A \) such that \( \xi_x(A) \neq \xi_y(A) \). Therefore \( \xi_x \neq \xi_y \).

To show that \( \eta_X \) is surjective we choose a point \( \xi \), and consider the closure space \( X \), let \( X \). To see the naturality of \( \eta \), consider continuous \( f : X \to Y \) where \( X \) and \( Y \) are complete \( T_0 \) closure spaces. We have the following compositions: 

\[
(S(O(f)) \circ \eta_X)(x) = (S(f^{-1}))(\xi_x) = \xi_x \circ f^{-1} = \xi_f(x) \text{ and } \eta_Y \circ f(x) = \xi_f(x).
\]

Hence \( \eta = (\eta_X)_X \in CLat_{\top} \) is a natural isomorphism \( \eta : 1_{CLat_{\top}} \simeq SO \).

The naturality of \( \epsilon \) follows since if \( h : \mathcal{L} \to \mathcal{L}' \) is a \( CLat_{\top} \)-morphism, we have the compositions 

\[
(\epsilon_{\mathcal{L}'} \circ O(h))(\Sigma_u) = \epsilon_{\mathcal{L}'}((S(h))^{-1}(\Sigma_u)) = (\epsilon_{\mathcal{L}'}(\Sigma_u))(\xi \circ h(u)) = \xi' h(u).
\]

Hence we have proven the above equivalence.

\[\square\]

**Example 3.1.11.** [A closure space which illustrates the above duality.]

We consider the closure space \( X \) on \( X = \{a, b, c, d, e\} \) given by the closed sets \( \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{d\}\} \) (see Figure 3.1). It is a \( T_0 \) closure space, which is not \( T_1 \). The open sets are \( \{\{d, e\}, \{c, d, b, e\}, \{c, d, e\}, \{c, b, e, a\}, \{c, d, b, e, a\}, \emptyset\} \).
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Figure 3.1: The basic closed sets of the closure space from Example 3.1.11.

and the lattice of open sets is given in Figure 3.2, where:

\begin{align*}
1 & = \emptyset \\
2 & = \{d, e\} \\
3 & = \{c, d, e\} \\
4 & = \{c, b, e, a\} \\
5 & = \{c, d, b, e\} \\
6 & = \{c, d, b, e, a\}
\end{align*}
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The points of this lattice are the following:

\[ \xi_1 = I_{\{4,6\}} \]
\[ \xi_2 = I_{\{2,3,5,6\}} \]
\[ \xi_3 = I_{\{3,4,5,6\}} \]
\[ \xi_4 = I_{\{4,5,6\}} \]
\[ \xi_5 = I_{\{2,3,4,5,6\}} \]

where \( I_A(a) = \{ \top \quad a \in A \quad \bot \quad a \notin A \} \). The closure space on the set of these points given by the open sets \( \Sigma_u, u \in \{1,2,3,4,5,6\} \) is isomorphic to \( X \), since each of these points \( \xi \) corresponds an \( O \)-stack stack \( \xi^{-1}(\top) \) of \( X \) which is the neighborhood stack of unique point:

\[ A_1 = \text{stack } \{ \{c, b, e, a\}, \{c, d, b, e, a\}\} = \mathcal{V}_X(a) \]
\[ A_2 = \text{stack } \{ \{d, e\}, \{c, d, b, e\}, \{c, d, e\}, \{c, d, b, e, a\}\} = \mathcal{V}_X(d) \]
\[ A_3 = \text{stack } \{ \{c, d, b, e\}, \{c, d, e\}, \{c, b, e, a\}, \{c, d, b, e, a\}\} = \mathcal{V}_X(c) \]
\[ A_4 = \text{stack } \{ \{c, d, b, e\}, \{c, b, e, a\}, \{c, d, b, e, a\}\} = \mathcal{V}_X(b) \]
\[ A_5 = \text{stack } \{ \{d, e\}, \{c, d, b, e\}, \{c, d, e\}, \{c, b, e, a\}, \{c, d, b, e, a\}\} = \mathcal{V}_X(e) \]

Since every \( O \)-stack is a neighborhood stack, \( X \) is indeed a complete \( T_0 \) closure space.

**Remark 3.1.12.** The functors from Theorem 3.1.10 are formally the same as the well-known functors

\[ \Omega : \text{Sob} \to \text{SpFrm}^{op} \]
\[ \Sigma : \text{SpFrm}^{op} \to \text{Sob} \]

describing the dual equivalence between sober topological spaces and spatial frames. However on an object level the complete \( T_0 \) closure spaces can not be regarded as a generalization of sober topological spaces. Whereas in the topological case there are many sober spaces (every Hausdorff space is sober), the condition of being a point-closure space is much more restrictive (no non-trivial \( T_1 \) closure space can be a point-closure space).

The previous results were developed in collaboration with E. Giuli and E. Lowen-Colebunders and were published in [20].
3. Representations of closure spaces

3.2 Equivalences

As announced earlier we will now follow the idea’s of M. Ernė, in which one represents a closure space by means of the lattice of closed sets. However in [26] closure spaces are not assumed to be grounded. To ensure that the closures we obtain are grounded we will slightly restrict the notion of invariant selection.

For any complete lattice $L$ we will say that a subset $S$ of $L$ is a $\lor$-base if for every $a \in L$ there are $(s_i)_{i \in I}$ in $S$ such that $a = \lor_{i \in I} s_i$. We write $\text{CLat}_\lor$ for the category of complete lattices with $\lor$-preserving maps.

From M. Ernė’s work we have the following propositions.

**Proposition 3.2.1.** We have the following functor:

$$G : \text{Cl} \rightarrow \text{CLat}_\lor :$$

$$X \mapsto \mathcal{C}X$$

$$f \mapsto \bar{f}$$

where for $f : X \rightarrow Y$, $\bar{f}$ denotes $\bar{f} : \mathcal{C}X \rightarrow \mathcal{C}Y : C \mapsto \text{cl}_Y(f(C))$.

**Proposition 3.2.2.** If $C$ is a full iso-closed subcategory of $\text{Cl}$ then the restriction $G : C \rightarrow \text{CLat}_\lor$ is faithful if and only if $C$ consists of only $T_0$ closure spaces.

**Definition 3.2.3.** An invariant selection $\Sigma$ is an operation which assigns to each complete lattice $\mathcal{L}$ a subset $\Sigma(\mathcal{L})$ of $\mathcal{L}\setminus\{\bot\}$, such that for every $\text{CLat}_\lor$-isomorphism $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ one has that $\phi(\Sigma(\mathcal{L})) = \Sigma(\mathcal{L}')$.

**Remark 3.2.4.** Note that $\Sigma$ is a class-theoretic operation, later on we will generalize it so that it becomes a functor. The fact that we do not want $\bot$ to be in $\Sigma(\mathcal{L})$ results from our wish to keep groundedness for closure spaces.

One can choose $\Sigma(\mathcal{L})$ to be $\mathcal{L}\setminus\{\bot\}$ or to be the set $A_{\mathcal{L}}$ of atoms of $\mathcal{L}$. Another possible choice is the set $I_\lor(\mathcal{L})$ of $\lor$-irreducible elements of $\mathcal{L}$, where $a \in \mathcal{L}$ is called $\lor$-irreducible if for every finite $A \subset \mathcal{L}$, $a = \lor A$ implies $a \in A$.

With an invariant selection $\Sigma$ we will associate an iso-closed subcategory $L_\Sigma$ of $\text{CLat}_\lor$. The objects of $L_\Sigma$ are exactly those complete lattices $\mathcal{L}$ for which $\Sigma(\mathcal{L})$ is a $\lor$-base. The $L_\Sigma$-morphisms are the $\text{CLat}_\lor$-morphisms $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ such that $\phi(\Sigma(\mathcal{L})) \subset \Sigma(\mathcal{L}')$. 

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With $\Sigma$ we also associate a full iso-closed subcategory $\mathcal{C}_\Sigma$ of $\mathcal{Cl}_0$ by taking as objects the $T_0$ closure spaces $X$ for which $\Sigma(G(X)) = \{cl_X \{x\} | x \in X\}$, i.e. $\Sigma(G(X))$ is exactly the set of point-closures of $X$.

From [26] we now have the following propositions.

**Proposition 3.2.5.** For a fixed invariant selection $\Sigma$ the functor $G$ restricts as follows:

\[ G_\Sigma : \mathcal{C}_\Sigma \rightarrow \mathcal{L}_\Sigma \]

**Proposition 3.2.6.** The following defines a functor:

\[ H_\Sigma : \mathcal{L}_\Sigma \rightarrow \mathcal{C}_\Sigma : \]

\[ \mathcal{L} \mapsto (\Sigma(\mathcal{L}), \mathcal{C}_\Sigma(\mathcal{L})) \]

\[ \phi \mapsto \phi |_{\Sigma(\mathcal{L})} \]

where the $\mathcal{C}_\Sigma(\mathcal{L}) = \{ \{b \in \Sigma(\mathcal{L}) | b \leq a\} | a \in \mathcal{L}\}$ are the closed sets of a closure space and $\phi |_{\Sigma(\mathcal{L})}$ denotes the restriction of the morphism $\phi : \mathcal{L} \rightarrow \mathcal{L}'$.

**Proof.** We show that $H_\Sigma(\mathcal{L})$ is indeed a closure space. Since $\Sigma(\mathcal{L})$ does never contain $\perp$, we get that $\emptyset$ is closed. By the equation $\cap_{i \in I} \{ b \in \Sigma(\mathcal{L}) | b \leq a_i \} = \{ b \in \Sigma(\mathcal{L}) | b \leq \bigwedge_{i \in I} a_i \}$ we see that $\mathcal{C}_\Sigma(\mathcal{L})$ is indeed a closure structure. For any $\mathcal{L}_\Sigma$-morphism $\phi$, the restriction $\phi |_{\Sigma(\mathcal{L})}$ is a continuous function. For a more detailed proof we refer to [26].

Finally we obtain the main result of [26].

**Theorem 3.2.7.** For an invariant selection $\Sigma$ the functors:

\[ G_\Sigma : \mathcal{C}_\Sigma \rightarrow \mathcal{L}_\Sigma \]

\[ H_\Sigma : \mathcal{L}_\Sigma \rightarrow \mathcal{C}_\Sigma \]

establish an equivalence between categories, by means of the natural isomorphisms:

\[ \epsilon_\mathcal{L} : G_\Sigma H_\Sigma(\mathcal{L}) \rightarrow \mathcal{L} : A \rightarrow \bigvee A \]

\[ \eta_X : X \rightarrow H_\Sigma G_\Sigma(X) : x \mapsto cl_X \{x\} \]
3. Representations of closure spaces

3.2.1 Making concrete equivalences

What we have done until now is recalling the theory outlined in [26]. In this section we will improve these results in order to get a concrete equivalence in the sense of H.-E. Porst [40].

A concrete isomorphism between two concrete categories is a well-defined notion [1], however in this book the authors state ([1], 5.13) that it makes little sense to talk about concrete equivalences between categories since there are equivalent concrete categories \( A \) and \( B \), where the equivalence \( F : A \to B \) is indeed a concrete functor but where there is no equivalence \( G : B \to A \) which is concrete. In [40] H.-E. Porst proposed another concept of concrete functor, which does allow to define concrete equivalences.

**Definition 3.2.8.** A concrete functor between two concrete categories \((A,U : A \to X)\) and \((B,V : B \to X)\) over \(X\) is a pair \((F,\phi)\) where \(F : A \to B\) is a functor and \(\phi : VF \to U\) is a natural isomorphism. Schematically we write this situation as

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow{\phi} & & \downarrow{\psi} \\
U & \xleftarrow{\phi^{-1}} & V \\
& X & \\
\end{array}
\]

Composition of two concrete functors \((F,\phi) : (A,U) \to (B,V)\) and \((G,\psi) : (B,V) \to (C,W)\) over \(X\) is defined as \((G,\psi) \circ (F,\phi) = (GF,\phi \circ \psi F)\).

**Definition 3.2.9.** Let \((F,\phi),(G,\psi) : (A,U) \to (B,V)\) be two concrete functors. A natural transformation \(\pi : F \to G\) is a concrete natural transformation if:

\[
VF \xrightarrow{\psi \circ \pi} VG \xleftarrow{\phi} U = VF \xrightarrow{\phi} U
\]

**Definition 3.2.10.** A concrete functor \((F,\phi) : (A,U) \to (B,V)\) is called a concrete equivalence if there exists a concrete functor \((G,\psi) : (B,V) \to (A,U)\) and concrete natural isomorphisms

\[
\zeta : (G,\psi) \circ (F,\phi) \to (1_A,1_U) \quad \text{and} \quad \chi : (F,\phi) \circ (G,\psi) \to (1_B,1_V)
\]
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We now return to the equivalence between $C_\Sigma$ and $L_\Sigma$. Since $C_\Sigma$ is a full subcategory of the construct $(\mathbf{Cl}, U)$, the forgetful functor can be restricted. In this way we get the construct $(C_\Sigma, U)$. Keeping the concept of concrete equivalence in mind, we define a forgetful functor on $L_\Sigma$ in the following way:

$$U_\Sigma : L_\Sigma \rightarrow \textbf{Set} : \begin{array}{c}
\mathcal{L} \\
\phi
\end{array} \mapsto \begin{array}{c}
\Sigma(\mathcal{L}) \\
\phi|_{\Sigma(\mathcal{L})}
\end{array}$$

By the fact that $\Sigma(\mathcal{L})$ is a $\lor$-base for $\mathcal{L}$ we have that $U_\Sigma$ is indeed faithful. Hence $(L_\Sigma, U_\Sigma)$ is a construct. In order to produce a concrete functor we need the following natural isomorphisms:

$$\begin{array}{c}
\phi : U_\Sigma G_\Sigma \rightarrow U \\
\psi : U_\Sigma H_\Sigma \rightarrow U_\Sigma
\end{array}$$

defined by the $\textbf{Set}$-isomorphisms

$$\begin{array}{c}
\phi_X : \Sigma(G_\Sigma(X)) \rightarrow X : d_X(\{x\}) \mapsto x \\
\psi_\mathcal{L} : \Sigma(\mathcal{L}) \rightarrow \Sigma(\mathcal{L}) : a \mapsto a
\end{array}$$

Note that $\phi_X$ is a bijection since $X$ is a $C_\Sigma$-object and as such it is $T_0$.

**Proposition 3.2.11.**

$$(G_\Sigma, \phi) : (C_\Sigma, U) \rightarrow (L_\Sigma, U_\Sigma) \text{ and } (H_\Sigma, \psi) : (L_\Sigma, U_\Sigma) \rightarrow (C_\Sigma, U)$$

are concrete functors, i.e.

$$(\xymatrix{C_\Sigma \ar[r]^{G_\Sigma} & L_\Sigma \\
\text{Set} \ar[ur]_{U_\Sigma} \ar[r]_{\phi} & \Sigma(\mathcal{L})} \quad \quad (\xymatrix{L_\Sigma \ar[r]^{H_\Sigma} & C_\Sigma \\
\text{Set} \ar[ur]_{U_\Sigma} \ar[r]_{\psi} & \Sigma(\mathcal{L})})$$

**Theorem 3.2.12.** The concrete functors

$$(G_\Sigma, \phi) : (C_\Sigma, U) \rightarrow (L_\Sigma, U_\Sigma) \text{ and } (H_\Sigma, \psi) : (L_\Sigma, U_\Sigma) \rightarrow (C_\Sigma, U)$$

form a concrete equivalence.

**Proof.** We have two show the existence of two concrete natural isomorphisms

$$\begin{array}{c}
\zeta : (H_\Sigma, \psi) \circ (G_\Sigma, \phi) \rightarrow (1_{C_\Sigma}, 1_U) \\
\chi : (G_\Sigma, \phi) \circ (H_\Sigma, \psi) \rightarrow (1_{L_\Sigma}, 1_{U_\Sigma})
\end{array}$$
Since \((H_\Sigma, \psi) \circ (G_\Sigma, \phi) = (H_\Sigma G_\Sigma, \phi \circ \psi G_\Sigma)\) and \((G_\Sigma, \phi) \circ (H_\Sigma, \psi) = (G_\Sigma H_\Sigma, \psi \circ \phi H_\Sigma)\) we define \(\zeta\) and \(\chi\) by means of \(\zeta_X = \eta_X^{-1}\) and \(\chi_L = \eta_L^{-1} = \epsilon_L\), were \(\eta\) and \(\epsilon\) are the natural isomorphisms introduced by the main theorem of [26] (see Theorem 3.2.7). It remains to prove that

\[ \zeta : (H_\Sigma, \psi) \circ (G_\Sigma, \phi) \to (1 \Sigma, 1_U) \quad \text{and} \quad \chi : (G_\Sigma, \phi) \circ (H_\Sigma, \psi) \to (1 \Sigma, 1_U) \]

are concrete. So we need to show that

\[ U H_\Sigma G_\Sigma \xrightarrow{U \zeta} U 1 \Sigma \xrightarrow{1_U} U = U H_\Sigma G_\Sigma \xrightarrow{\phi \circ \psi G_\Sigma} U \]

For any \(C_\Sigma\)-object \(X\) we have the \text{Set}-morphism

\[ (1_U \circ U \zeta)_X = (1_U)_X \circ U \zeta_X : U H_\Sigma G_\Sigma X \to U X : d(\{x\}) \mapsto \eta_X^{-1}(d(\{x\})) = x \]

On the other side of the equation one has:

\[ (\phi \circ \psi G_\Sigma)_X = \phi_X \circ \psi G_\Sigma X : U H_\Sigma G_\Sigma X \to U X : d(\{x\}) \mapsto \phi_X \circ \psi G_\Sigma X(d(\{x\})) = x \]

Hence \(\zeta\) is concrete. For \(\chi\) one needs to consider the concrete functors

\[ (G_\Sigma H_\Sigma, \psi \circ \phi H_\Sigma), (1 \Sigma, 1_U) : (L_\Sigma, U_\Sigma) \to (L_\Sigma, U_\Sigma) \]

in order to prove that

\[ U \Sigma G_\Sigma H_\Sigma \xrightarrow{U \Sigma \chi} U \Sigma 1 \Sigma \xrightarrow{1_U \Sigma} U \Sigma = U \Sigma G_\Sigma H_\Sigma \xrightarrow{\psi \circ \phi H_\Sigma} U \Sigma \]

For any \(L_\Sigma\)-object \(L\) we have the \text{Set}-morphism

\[ (1_U \circ U \Sigma \chi)_L = (1_U)_L \circ U \Sigma \chi_L : U \Sigma G_\Sigma H_\Sigma L \to U \Sigma L : \]

\[ A = \{b \in \Sigma(L) \mid b \leq a\} \mapsto \epsilon_L(A) = a \]

where \(a \in \Sigma(L)\). On the other side of the equation one has:

\[ (\psi \circ \phi H_\Sigma)_L = \psi_L \circ \phi H_\Sigma L : U \Sigma G_\Sigma H_\Sigma L \to U \Sigma L : \]

\[ A = \{b \in \Sigma(L) \mid b \leq a\} \mapsto \psi_L \circ \phi H_\Sigma L(A) = a \]

Hence \(\chi\) is concrete.
3. Representations of closure spaces

3.2.2 Special cases

In [26] many examples of the previously described equivalences are given. Each of them can be seen as a concrete equivalence by making the construct $(L_\Sigma, U_\Sigma)$. We will recall three special cases.

Example 3.2.13. [The equivalence of $\text{Cl}_1$ and $\text{CALat}$]
Let $\Sigma$ be the invariant selection mapping $L$ to its atoms, $\Sigma(L) = A_L$. The lattices in $L_\Sigma$ are the complete atomistic lattices and the morphisms are the $\lor$-preserving maps which also preserve atoms, we will write this category of complete atomistic lattices as $\text{CALat}$. The category $C_\Sigma$ is given by those $T_0$ closure spaces $X$ for which $\{\text{cl}_X(x)|x \in X\}$ are atoms in the lattice of closed sets. One has that $y \in \text{cl}_X(x)$ implies $\text{cl}_X(y) \cap \text{cl}_X(x) \neq \emptyset$ so $\text{cl}_X(y) = \text{cl}_X(x)$ and by the $T_0$ property $y = x$. Therefore $\text{cl}_X(x) = \{x\}$ and $C_\Sigma$ is the category $\text{Cl}_1$ of $T_1$ closure spaces. With the underlying functor $A = U_\Sigma : \text{CALat} \to \text{Set} : L \mapsto A_L$, we get the concrete equivalence:

$$
\begin{array}{ccc}
\text{Cl}_1 & \xrightarrow{G_\Sigma} & \text{CALat} \\
H_\Sigma & \downarrow & \\
\downarrow & \text{Set} & \downarrow \\
\phi & \downarrow & \psi \\
\downarrow & \text{CALat} & \downarrow \\
\downarrow & \text{Set} & \downarrow \\
\phi & \downarrow & \psi
\end{array}
$$

This equivalence is well-known and has been studied in [27], [28]. It also formed the main underlying idea of the equivalence between atomistic state property systems and $T_1$ closure spaces, which we have mentioned in the first chapter (Example 1.3.18).

Example 3.2.14. [The case of $\text{SobCl}$]
Next we choose $\Sigma(L)$ to be the set $I_\lor(L)$ of all $\lor$-irreducible elements of $L$. In this case $L_\Sigma$ consists of so-called $\lor$-decomposition lattices [26] with suitable morphisms. The closure spaces of $C_\Sigma$ are the ones for which the set of point-closures is exactly the set of $\lor$-irreducible closed sets. Inspired by the analogous property of sober topological spaces, M. Erné calls these sober closure spaces, the corresponding category of sober closure spaces will be written as $\text{SobCl}$. Although these space are a generalization of sober topological spaces we know that they do not occupy the same place within the $T_0$ closure spaces as their topological counterparts, since this place is already occupied by the point-closure spaces. With the underlying functor $U_\Sigma : L_\Sigma \to \text{Set} : L \mapsto I_\lor(L)$,
we get the concrete equivalence:

\[
\begin{array}{c}
\text{SobCl} \\ U \\
\end{array} \xrightarrow{\phi} \begin{array}{c}
\Sigma \\ U' \\
\end{array} \xrightarrow{\psi} \begin{array}{c}
\text{Lat}^* \\ \Sigma \\
\end{array}
\]

\[
\begin{array}{c}
\text{Set} \\
\end{array}
\]

Example 3.2.15. [The equivalence between \(\text{CCl}_0\) and \(\text{CLat}_\lor^*\)]

Our last example is produced by taking \(\Sigma(\mathcal{L}) = \mathcal{L}\setminus\{\bot\}\). In this case the closure spaces of \(\mathcal{C}_\Sigma\) are the ones where every nonempty closed set is a point-closure, i.e. the point-closure spaces \(\mathcal{C}_\Sigma = \text{CCl}_0\). The category \(\text{L}_\Sigma\) consists of all complete lattices and the morphisms are the \(\lor\)-preserving maps \(\phi : \mathcal{L} \to \mathcal{L}'\) for which \(\phi(\Sigma(\mathcal{L})) \subseteq \Sigma(\mathcal{L}')\), i.e. the ones for which \(\phi(\bot) = \bot'\) implies \(\mathcal{L} = \bot\). Writing \(\text{CLat}_\lor^*\) for this category, we have the following concrete equivalence:

\[
\begin{array}{c}
\text{CCl}_0 \\ U \\
\end{array} \xrightarrow{\phi} \begin{array}{c}
\text{CLat}^* \\ U' \\
\end{array} \xrightarrow{\psi} \begin{array}{c}
\text{Set} \\
\end{array}
\]

Where \(U' : \text{CLat}_\lor^* \to \text{Set}\) is the underlying functor, given by \(U'(\mathcal{L}) = \mathcal{L}\setminus\{\bot\}\).

Thus we have a dual equivalence between \(\text{CLat}_{\lor,\top}\) and \(\text{CCl}_0\). And there is an equivalence between \(\text{CLat}_{\lor}^*\) and \(\text{CCl}_0\). Combining them we get a dual equivalence between the categories \(\text{CLat}_{\lor}^*\) and \(\text{CLat}_{\lor,\top}\) which is given by the functor:

\[
D : \text{CLat}_{\lor}^* \to \text{CLat}_{\lor,\top}^{\text{op}}
\]

\[
\mathcal{L} \mapsto \mathcal{L}^d
\]

\[
\phi \mapsto \phi^{-1}
\]

(here \(\mathcal{L}^d\) is the dual lattice of \(\mathcal{L}\)).

3.3 A Stone-like duality

The well-known Stone representation theorem gives a dual equivalence between the category of Boolean algebras and the category of compact zero-dimensional
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Hausdorff topological spaces [44]. In what follows we will give an analogue to this representation theorem based on the work of R. Mayet [35]. We introduce a suitable form of compactness for zero-dimensional closure spaces, using convergence of certain stacks. With this concept of compactness we will obtain a dual equivalence between the category of complemented partial ordered sets and the category of compact zero-dimensional $T_0$ closure spaces.

We start by noticing that for a closure space $X$ the collection $\text{CO}(X)$ of clopen subsets of $X$ does not always form a complete lattice, instead it is only a partially ordered set, with a notion of complementation.

Example 3.3.1. [A 0Cl-space $X$ such that $\text{CO}(X)$ is not a complete lattice.]
Let us take $X = \{1, 2, 3, 4, 5\}$ and define

$\begin{align*}
A_1 &= \{1\} & B_1 &= \{1, 2, 3, 4\} \\
A_2 &= \{2\} & B_2 &= \{1, 2, 3, 5\}
\end{align*}$

Then we choose $A_1, A_2, B_1, B_2$ and their complements as a base for the open sets of the zero-dimensional closure space $X$. In this case the clopen sets of $X$ are $\text{CO}(X) = \{X, \emptyset, A_1, A_2, B_1, B_2, X \setminus A_1, X \setminus A_2, X \setminus B_1, X \setminus B_2\}$. One sees that $A_1 \lor A_2$ does not exist in the partially ordered set $(\text{CO}(X), \subset)$. ◀

This observation leads to the definition of the following category.

Definition 3.3.2. $P = (P, \leq^\triangledown)$ is a complemented partially ordered set (complemented poset) if $(P, \leq)$ is a partially ordered set with a maximum $\top$, a minimum $\bot$ and, a complementation $^\triangledown : P \to P$ such that:

1. $\forall a \in P : (a^\triangledown)^\triangledown = a$
2. $\forall a, b \in P : a \leq b \Rightarrow b^\triangledown \leq a^\triangledown$
3. $\forall a \in P : a \wedge a^\triangledown = \bot$ and $a \lor a^\triangledown = \top$

A morphism between two complemented posets $P, Q$ is a monotone map $f : P \to Q$ which preserves complementation, i.e. $\forall a \in P : f(a^\triangledown) = (f(a))^\triangledown$. In this way we get the category $^\triangledown\text{PO}$ of complemented posets. ◀

The notion of complemented poset, has been extensively investigated in the context of quantum logic, where it sometimes is referred to as an orthocomplemented poset or orthoposet [55]. It is known that $^\triangledown\text{PO}$ is dually equivalent...
3. Representations of closure spaces

with a subcategory of Cl, which consists of R. Mayet’s “C-espaces” [35]. We will introduce a suitable notion of compactness which will enable us to characterize these “C-espaces” as zero-dimensional Hausdorff closure spaces which are compact in the sense that certain prime stacks must converge.

In the classical case of a Boolean algebra one considers prime filters or equivalently maximal filters (ultrafilters) of the Boolean algebra in order to construct a topological space. Such a prime filter of a Boolean algebra $B$ is proper filter $F$ for which $\forall a \in B : a \in F$ or $a^\complement \in F$, since $F$ is proper it can not be that both $a$ and $a^\complement$ are in $F$. We now define prime stacks.

**Definition 3.3.3.** Let $P$ be a poset. A subset $A \subset P$ is called a stack of $P$ if $a \in A, a \leq b$ implies $b \in A$. We will write $\dot{a}$ for the stack \{ $b \in P | a \leq b$ \}. A prime stack on an complemented poset $P$ is a stack $A$ such that for any $a \in P$ either $a \in A$ or $a^\complement \in A$, but not both. ◀

Prime stacks are obviously not maximal stacks. However they are the maximal elements of the collection

$$\chi(P) = \{ A \text{ stack in } P | \exists a \in P : a \in A \text{ and } a^\complement \in A \}$$

The stacks in $\chi(P)$ are called “orthosections finales” in [35].

In any Boolean algebra, a prime filter is a prime stack. For a closure space $X$ and a point $x \in X$ the collection $V_{c,X}(x) = \{ A \in CO(X) | x \in A \}$ is a prime stack of $CO(X)$.

**Definition 3.3.4.** Let $X$ be a zero-dimensional closure space. A collection $A \subset P(X)$ is called a prime stack of $X$ if it is a prime stack of $(CO(X), \subset)$, i.e. a maximal element of $\chi_{CO(X)}$. $X$ is called stack compact or s-compact if every prime stack $A$ converges, i.e. there is a $x \in X$ such that $V_{c,X}(x) = A$.

The full subcategory of $0Cl_0$ consisting of s-compact objects will be written as $SComp0Cl_0$. ◀

**Remark 3.3.5.** We note here that if a topological space $X$ is s-compact as a closure space, then it is compact in the usual sense, since every ultrafilter is an prime stack. However compactness and s-compactness are not equivalent as is shown in the next example. ◀

**Example 3.3.6.** [A topological space which is compact but not s-compact.]

Consider the discrete topological space $D_3$ on three points $\{1, 2, 3\}$. Since it is a finite space, it is compact. The prime stack $A = \{ \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\} \}$ does not converge, so this space is not s-compact. The only other prime stacks are $\dot{1}, \dot{2}, \dot{3}$. ◀
3. Representations of closure spaces

Let $P$ be a complemented poset. As in [35] we define a closure space $S(P)$ on the set $S_P$ of all prime stacks of $P$, by using the collection $\{A_a | a \in P\}$ as a base for the open sets of $S(P)$ (here we write $A_a$ for the set $\{A$ prime stack of $P$ with $a \in A\}$).

Lemma 3.3.7. Let $B$ clopen in $S(P)$, then there exists an $a \in P$ such that $B = A_a$.

Proof. Let $B$ be a proper clopen subset of $S(P)$ (in the other case the lemma clearly holds). Suppose $B \neq A_a$ for all $a \in P$. We consider $A_B = \{b \in P | A_B \subseteq S_P \setminus B\}$, which are two nonempty sets. Let $A$ be the stack $\{a^B | a \in A \cup A_B\}$, then $A \in \chi_P$. If not, there would be some $a \in P$ such that $a, a^B \in A \cup A_B$, but this would imply that $B$ is one of $A_a$ or $A_a^B$, contradicting our choice of $B$. Hence there exists a prime stack $A^* \subseteq A$. Since $B$ is a clopen subset of $S(P)$ there is an element $b \in P$ such that $A^* \subseteq A_B \subseteq B$. But this implies that $b$ and $b^B$ are in $A^*$ which is impossible, therefore $B = A_a$ for some $a \in P$. 

We have the following proposition.

Proposition 3.3.8. Let $P$ be a complemented poset then $S(P)$ is a s-compact zero-dimensional Hausdorff closure space.

Proof. That $S(P)$ is a zero-dimensional closure space follows from the fact that $S_P \setminus A_a = A_a^\varnothing$. It is Hausdorff since for any two different prime stacks $A, B$ one has an $a \in P$ such that $a \in A$ and $a \notin B$. In order to prove s-compactness we choose a prime stack $A$ on $S(B)$. For each $a \in P$ one has either $A_a \in A$ or $A_a^\varnothing \in A$. For each $a \in P$ we choose $b_a$ such that $A_{b_a} \in A$. In this way we obtain a prime stack $\{b_a | a \in P\}$ in $P$ which is a point of $S(P)$. Writing $p$ for this point, one obtains by construction and in view of the preceding lemma, that $\forall_c \mathcal{X}(p) = A$.

Using these results R. Mayet’s dual equivalence can be written as follows.

Theorem 3.3.9. The functors

\[
\begin{align*}
S : \mathcal{CPO}^{op} & \rightarrow \mathbb{SComp0Cl}_0 \\
P & \mapsto S(P) \\
B : \mathbb{SComp0Cl}_0 & \rightarrow \mathcal{CPO}^{op} \\
X & \mapsto \text{CO}(X) \\
f : X \rightarrow Y & \mapsto B(f)
\end{align*}
\]
where for a prime stack \( A \) of \( \mathbb{P} \) and the \( \mathcal{PO} \) morphism \( f : Q \to P \), \( f^{-1}(A) = \cup_{a \in A} f^{-1}(a) \) and \( B(f) \) is given by the \( \mathcal{PO} \) morphism \( f^{-1} : CO(Y) \to CO(X) \), define an equivalence by means of the natural isomorphisms \( \eta \) and \( \epsilon \) for which

\[
\eta_P : P \to CO(S(P)) : a \mapsto A_a
\]

and

\[
\epsilon_X : X \to S(CO(X)) : x \mapsto V_{c,X}(x)
\]

We have however seen that \( s \)-compactness does not reduce to the usual compactness in the case of a topological space, hence as is observed in [35] the dual equivalence described here does not reduce to the Stone duality.

**Theorem 3.3.10.** The category \( \text{SComp0Cl}_0 \) is an epireflective subcategory of \( \text{0Cl}_0 \). The epireflection of a zero-dimensional \( T_0 \) closure space \( X \) is given by the embedding \( r : X \to SBX : x \mapsto V_{c,X}(x) \).

**Proof.** A basic open set of \( SBX \) is of the form \( A_B \), where \( B \) is a clopen set of \( X \). Since \( r^{-1}(AB) = \{ x \in X | B \in V_{c,X}(x) \} = B \), we obtain that \( r \) is an initial morphism. Because \( X \) is \( T_0 \), \( r \) is an embedding.

To see that \( r \) is an epimorphism we consider two morphisms \( h, k : SBX \to Y \) to a \( \text{0Cl}_0 \)-object \( Y \) such that \( h \circ r = k \circ r \). Suppose that \( h \neq k \) then there is a prime stack \( A \) of \( X \) such that \( h(A) \) and \( k(A) \) are different. Since \( Y \) is Hausdorff and zero-dimensional there is a clopen subset \( B \) of \( Y \) such that \( h(A) \in B \) and \( k(A) \in Y \setminus B \). By Lemma 3.3.7 we know that there exist \( B_1, B_2 \in CO(X) \) for which \( A \in h^{-1}(B) = B_{A_1} \) and \( A \in k^{-1}(Y \setminus B) = A_{B_2} \). From the first part of this proof we know that \( r^{-1}(A_{B_1}) = B_1 \) and \( r^{-1}(A_{B_2}) = B_2 \). Since \( h \circ r = k \circ r \) we find that \( B_1 \) and \( B_2 \) are complementary clopen sets of \( X \) which are both contained in \( A \). This yields a contradiction with the primality of \( A \), hence \( h = k \).

Consider a continuous map \( f : X \to Y \) to a \( \text{SComp0Cl}_0 \)-object and define for any \( A \in S_{CO(Y)} : B(A) = \{ B \in CO(Y) | \exists A \in A | f(A) \subset B \} \). For any \( A \in CO(Y) \) we have that \( f^{-1}(A) \) is clopen in \( X \), hence either one of \( f^{-1}(A) \) or \( f^{-1}(X \setminus A) \) is in \( A \), so \( A \in B(A) \) or \( X \setminus A \in B(A) \). If both are in \( B(A) \) this would imply that both \( f^{-1}(A) \) and \( f^{-1}(X \setminus A) \) are in \( A \), which is impossible since \( A \) is a prime stack. Therefore \( B(A) \) is a prime stack of \( Y \). Since \( Y \) is \( s \)-compact one sees that \( f^* : SBX \to Y : A \mapsto \lim B(A) \) is a well-defined map. \( f^* \) is continuous because for \( B \in CO(Y) \), \( (f^*)^{-1}(B) = \{ A \in S_{CO(X)} | \exists A \in A | f(A) \subset B \} = A_{f^{-1}(B)} \).
Moreover we have $f^*(V_{c,X}(x)) = \lim \{ B \in CO(Y) | \exists A \in V _{c,X}(x) | f(A) \subset B \} = \lim V_{c,Y}(f(x)) = f(x)$. Uniqueness of $f^*$ follows since $r$ is an epimorphism, so $r : X \rightarrow SBX$ is the reflection:

$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
r & \downarrow & f^* \\
SBX & \xleftarrow{f^*} & \\
\end{array}$

From the theorem above it follows that s-compactness is productive and hereditary. It also follows that every zero-dimensional $T_0$ closure space possesses a unique s-compactification.

**Example 3.3.11.** [The s-compactification of $D_3$.]

The reflection of the space $D_3$ from Example 3.3.6 is the closure space on four points \{1, 2, 3, A\} with \{\{1\}, \{2\}, \{3\}, \{1, A\}, \{2, A\}, \{3, A\}\} as a base for the open sets. Remark that this is not a topological space.
Chapter 4

Completeness

In this chapter we will pay attention to the concept of completeness. When introducing this concept in a categorical sense one tries to copy the exemplary behavior of completeness in $\text{Unif}_0$. In this case the complete Hausdorff uniform spaces form a reflective subcategory, such that the reflection of a Hausdorff uniform space is a dense embedding, i.e. an epimorphic embedding. Moreover if a Hausdorff uniform space is densely embedded into two complete Hausdorff uniform spaces, they are necessarily isomorphic.

This has lead G.C.L. Brümmern and E. Giuli to the concept of firm $\mathcal{U}$-reflective subcategory [18] of a complete well-powered category $X$, which can be considered as the subcategory of complete objects of $X$. Such a subcategory is a reflective, full subcategory, for which the reflection $r_X : X \to RX$ of an $X$-object belongs to a given class of $X$-morphisms $\mathcal{U}$. Moreover one has that the “completion” is unique in the sense that a morphism $u : X \to Y$ is in $\mathcal{U}$ if and only if $Ru : RX \to RY$ is an isomorphism. If such a firm $\mathcal{U}$-reflective subcategory exists, it is unique and coincides with the full subcategory given by the $\mathcal{U}$-injective objects of $X$.

4.1 Firm $\mathcal{U}$-reflective subconstructs of $\Cl_0$

We will recall results from [18] and [19] in the case of a complete well-powered construct $X$ and with $\mathcal{U}$ being the class $\mathcal{U}_X$ of epimorphic embeddings of $X$. This will allow us to prove that the category $\CCl_0$ is the unique firm $\mathcal{U}_{\Cl_0}$-reflective subcategory of $\Cl_0$. We will also show that for $X = 0\Cl_0$ there is no firm $\mathcal{U}_{0\Cl_0}$-reflective subcategory.
4. Completeness

4.1.1 General theory

In this section we will use the following conventions. For any construct $X$ we will write $U_X$ for the class of epimorphic embeddings of $X$. We will assume that $X$ is a complete, well-powered construct with forgetful functor $U$. Subcategories will always be full and iso-closed. For a reflective subcategory $R$ of $X$ we will write $R$ for the reflection functor and $r_X : X \to RX$ for the reflection of an object $X$.

**Definition 4.1.1.** Following [18] we shall call a reflective subcategory $R$ of $X$ $U_X$-reflective if all $r_X$ are in $U_X$. A $U_X$-reflective subcategory $R$ of $X$ is then called a firm $U_X$-reflective subcategory whenever $Rf$ is an isomorphism if and only if $f \in U_X$.

**Remark 4.1.2.** Suppose $R$ is a firm $U_X$-reflective subcategory of $X$. Let $u_1 : X \to Y_1$ and $u_2 : X \to Y_2$ be epimorphic embeddings with codomain in $R$. By the firmness one has isomorphisms $Ru_1 : RX \to Y_1$ and $Ru_2 : RX \to Y_2$, hence $Y_1 \simeq Y_2$.

In the case $X = \text{Unif}_0$, $U_{\text{Unif}_0}$ are the dense embeddings. The subcategory $R = \text{CUnif}_0$ (the full subcategory of complete Hausdorff Uniform spaces) is firm $U_{\text{Unif}_0}$-reflective. The above definition and remark reduce to the well-known properties of completions of Hausdorff uniform spaces.

**Definition 4.1.3.** $J \in |X|$ is $U_X$-injective if and only if for all epimorphic embeddings $u : X \to Y$ and morphisms $f : X \to J$ in $X$ there is an $f' : Y \to J$ such that $f' \circ u = f$.

We shall write $\text{Inj}(U_X)$ for the full subcategory of $U_X$-injective objects of $X$.

**Theorem 4.1.4.** [18] Let $X$ be a complete well-powered construct. There exists at most one firm $U_X$-reflective subcategory $R$ of $X$. If such a subcategory exists we have $R = \text{Inj}(U_X)$.

**Definition 4.1.5.** Let $P$ be a class of $X$-objects. $X$ is $\text{Emb }X$-cogenerated by $P$ if every $X$-object is a subspace ($\text{Emb }X$-subobject) of a product of objects in $P$.

Combining this result with Theorem 1.6 of [19] we obtain the following.
Theorem 4.1.6. Let $X$ be a complete, well-powered construct, then $X$ admits a firm $U_X$-reflective subcategory $R$ if and only if there is a class $P$ consisting of $U_X$-injective objects which $Emb_X$-cogenerates $X$. In this case $R$ is the epireflective hull $E_X(P)$ of $P$ in $X$.

In the examples that will follow we will always be considering certain epireflective subconstructs $X$ of $Cl_0$ and we shall be interested in the existence of firm $U_X$-reflective subcategories. For this purpose we need some explicit description of the epimorphisms of $X$. We'll achieve this by using the so-called regular closure described by D. Dikranjan, E. Giuli and A. Tozzi in [24]. We will introduce the concept of regular closure in the more general context of closure operators on categories [23], [25].

We recall that for a construct $X$ with forgetful functor $U$ and $X \in |X|$, we write $M \subset X$ if we mean that $M$ is a subset of the underlying set $U(X)$ of $X$.

Definition 4.1.7. Let $X$ be a well-fibred topological construct. A closure operator on $X$ is a family
\[
\{c_X : \mathcal{P}(X) \to \mathcal{P}(X)\}_{X \in |X|}
\]
which satisfies:

1. $\forall M \subset X : M \subset c_X(M)$ (extensiveness)
2. $M \subset N \subset X \Rightarrow c_X(M) \subset c_X(N)$ (monotonicity)
3. $\forall X - \text{morphism } f : X \to Y : \forall M \subset X : f(c_X(M)) \subset c_Y(f(M))$ (continuity)

$c$ is said to be:

- grounded if $c_X(\emptyset) = \emptyset$
- idempotent if $c_X(c_X(M)) = c_X(M)$
- hereditary if $\forall M \subset X \subset Y : c_X(M) = c_Y(M) \cap X$
- weakly hereditary if $\forall M \subset X : c_X(M) = c_{c_X(M)}(M)$
Proposition 4.1.8. Let $X$ be a well-fibred topological construct with underlying functor $U$ and let $c$ be an idempotent, grounded closure operator on $X$. We have the following functor:

$$F_c : X \rightarrow \text{Cl}$$

$$X \mapsto (U(X), c_X)$$

$$f : X \rightarrow Y \mapsto F_c(f) : (U(X), c_X) \rightarrow (U(Y), c_Y)$$

Definition 4.1.9. Let $X$ be a well-fibred topological construct. Suppose $A$ is an (epi-)reflective subcategory of $X$. We write $\text{reg}^A = \{ \text{reg}^A_X \}_{X \in |X|}$ for the regular $A$-closure as defined in [24]. For $X \in |X|$ and $M \subset X$:

$$\text{reg}^A(M) = \{ x \in X | \forall f, g : X \rightarrow A \in |A| : f|M = g|M \Rightarrow f(x) = g(x) \}$$

We obtain the following well-known property.

Proposition 4.1.10. [24] Let $X$ be a well-fibred topological construct. Suppose $A$ is an (epi-)reflective subcategory of $X$. An $A$-morphism $f : X \rightarrow Y$ is an $A$-epimorphism if and only if it is $\text{reg}^A$-dense, i.e. $\text{reg}^A_Y(f(X)) = Y$.

Using Theorem 2.6 of [24] we know that in the case were $A$ is extremal epireflective in $X$, we can also get some information about the regular and extremal monomorphisms using the $A$-closure.

Proposition 4.1.11. Let $X$ be a well-fibred topological construct an let $A$ be an extremal epireflective subcategory of $X$ and $f : X \rightarrow Y$ any $A$-morphism. Then we have the following implications:

$f$ is an $\text{reg}^A$-closed embedding $\Rightarrow f$ is a regular monomorphism in $A$ $\Rightarrow f$ is an extremal monomorphism in $A$.

We are now ready to consider our first example, the category $\text{Cl}_0$ of $T_0$ closure spaces.

### 4.1.2 CCl$_0$ is the firm $\mathcal{U}_{\text{Cl}_0}$-reflective subcategory of $\text{Cl}_0$

In order to show that CCl$_0$ is the firm $\mathcal{U}_{\text{Cl}_0}$-reflective subcategory of $\text{Cl}_0$ we would like to use Theorem 4.1.6.
Remark 4.1.12. As we mentioned earlier, $\mathbf{Cl}_0$ is extremally epireflective in $\mathbf{Cl}$, as such it follows that $\mathbf{Cl}_0$ is an initially structured category in the sense of [37], thus it is a complete well-powered construct. The embeddings $\text{Emb} \mathbf{Cl}_0$ of $\mathbf{Cl}_0$ are the ones from $\mathbf{Cl}$. Bearing in mind that, in order to apply Theorem 4.1.6, we have to find some cogenerators for this category, we consider the following proposition.

Proposition 4.1.13. The $T_0$ closure spaces are exactly the subspaces of powers of the Sierpinski space $\mathbb{S}_2$.

Proof. Clearly any subspace of a power of $\mathbb{S}_2$ is $T_0$. Conversely, let $X$ be a $T_0$ closure space. Consider the following source:

$$(f : X \to \mathbb{S}_2)_{f \in C(X, \mathbb{S}_2)}$$

where $C(X, \mathbb{S}_2)$ denotes the set of all continuous functions from $X$ to $\mathbb{S}_2$. This source is initial, since if $A$ is an open set of $X$ we can consider the indicator $I_A : X \to \mathbb{S}_2$, which is continuous. Hence $A$ is open in the initial structure $\mathcal{O}_n$ and $\mathcal{O}_X \subset \mathcal{O}_n$. Conversely, we clearly have $\mathcal{O}_n \subset \mathcal{O}_X$. The above source is also point-separating since for $x \neq y$ there exist an open $V$ which separates $x$ and $y$, let us assume that $x \in V$ and $y \not\in V$, then $I_V(x) \neq I_V(y)$. We now have the following embedding:

$$i : X \to S^{C(X, \mathbb{S}_2)} : x \mapsto (f(x))_{f \in C(X, \mathbb{S}_2)}$$

Hence $X$ is a subspace of a power of $\mathbb{S}_2$. □

Now that we have a candidate $\mathbb{S}_2$ for cogenerator, the next thing we need is the regular $\mathbf{Cl}_0$-closure in order to characterize the $\mathbf{Cl}_0$-epimorphisms.

By definition the regular $\mathbf{Cl}_0$-closure is given as follows. For $X \in |\mathbf{Cl}_0|$ and $A \subset X$:

$$x \in \text{reg}_{\mathbf{Cl}_0}(A) \iff \forall f, g : X \to |\mathbf{Cl}_0| : f|_A = g|_A \Rightarrow f(x) = g(x)$$

Hence it coincides with the Zariski-closure as defined in [22]:

$$x \in z_X(A) \iff \forall u, v : X \to \mathbb{S}_2 : A \subset \{u = v\} \Rightarrow u(x) = v(x)$$

Definition 4.1.14. For any closure space $X$ we define the front-closure or $b$-closure, which is formally the same as in [23]. For $A \subset X$:

$$x \in b_X(A) \iff \forall \text{ open } U \in \mathcal{V}_X(x) : U \cap d_X \{x\} \cap A \neq \emptyset$$
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As in the topological case the $b$-closure is idempotent and hereditary, hence it is also weakly hereditary.

**Proposition 4.1.15.** $\text{reg}^{\text{Cl}_0} = z = b$

**Proof.** Let $X$ be a $T_0$ closure space and $A \subset X$. For $x \notin z_X(A)$ there are $u, v : X \to S_2$ with $A \subset u = v$ and $u(x) = 1, v(x) = 0$. Since both maps are continuous one has $\emptyset = u^{-1}(1) \cap v^{-1}(0) \cap A \supset u^{-1}(1) \cap d_X \{x\} \cap A$ hence $x \notin b_X(A)$. Conversely if $x \notin b_X(A)$ then there is an open $U \in \mathcal{V}(x)$ such that $U \cap d_X \{x\} \cap A = \emptyset$. Then $u = I_{X \backslash d_X \{x\}}$ and $v = I_{X \backslash d_X \{x\} \cup U}$ are continuous maps $X \to S_2$ such that $A \subset \{u = v\}$ and $u(x) \neq v(x)$, thus $x \notin z_X(A)$. $\square$

**Corollary 4.1.16.** The epimorphisms of $\text{Cl}_0$ are exactly the $b$-dense maps.

Since $U_{\text{Cl}_0}$ consists of all $b$-dense embeddings we can show the following.

**Proposition 4.1.17.** $S_2$ is $U_{\text{Cl}_0}$-injective in $\text{Cl}_0$.

**Proof.** Consider the following diagram of morphisms:

$$
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{f'} \\
S_2 & & \\
\end{array}
$$

where $u$ is an epimorphic embedding in $\text{Cl}_0$. $G = f'^{-1}(\{1\})$ is open in $X$, hence by the initiality of $u$ there exist an open $G'$ in $Y$ such that $G = u^{-1}(G')$. Define $f' = I_{G'}$, then $f'$ is a continuous extension of $f$ along $u$. $\square$

Because $S_2$ is $U_{\text{Cl}_0}$-injective and it $\text{Emb} \ \text{Cl}_0$-cogenerates $\text{Cl}_0$ and by the observations of Remark 4.1.12, we can apply Theorem 4.1.6 and we get the following proposition.

**Proposition 4.1.18.** $\text{Cl}_0$ admits a firm $U_{\text{Cl}_0}$-reflective subcategory, given by the epireflective hull $E_{\text{Cl}_0}(\{S_2\})$ in $\text{Cl}_0$.

We will show that $E_{\text{Cl}_0}(\{S_2\}) = C\text{Cl}_0$, i.e. the complete $T_0$ closure spaces we introduced in Definition 3.1.7 do behave as the complete objects of $\text{Cl}_0$. To achieve this result we will use the equivalence obtained in Theorem 3.1.10 in order to show that $C\text{Cl}_0$ is epireflective in $\text{Cl}_0$.  

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Proposition 4.1.19. Let $X^S$ denote the closure space on the set of $O$-stacks of a $T_0$ closure space $X$ given by the open sets $\{ A O$-stack $| A \in A \}, A \in O_X$. Then $r^S : X \rightarrow X^S : x \mapsto V_X(x)$ is the $CCl_0$-reflection of $X$.

Proof. To show this we construct the following isomorphism:

$$k : X^S \rightarrow SOX : A \mapsto \xi_A \quad k^{-1} : SOX \rightarrow X^S : \xi \mapsto \text{stack } \xi^{-1}(1)$$

where $\xi_A(A) = \begin{cases} 1 & A \in A \\ 0 & A \notin A \end{cases}$.

Proposition 4.1.20. $CCl_0$ is an epireflective subcategory of $Cl_0$.

Proof. We know that the epimorphisms of $Cl_0$ are the $b$-dense morphisms. Let $X$ be a $T_0$ closure space and consider the reflection: $r^S : X \rightarrow X^S$. Clearly $b_{X^S}(r^S(X)) \subset X^S$. We prove that $b_{X^S}(r^S(X)) \supset X^S$ also holds.

$$b_{X^S}(r^S(X)) = b_{X^S}(\{ V_X(x) | x \in X \}) = \{ A O$-stack $| \forall W \in V_A'(A) : A \notin \forall W \cap \{ V_X(x) | x \in X \} \neq \emptyset \}$$

Choose an arbitrary $O$-stack $A$ and an open neighborhood $W = \{ B O$-stack $| A \in B \}$ for some $A \in A$, open in $X$. Then it follows that $W \cap \{ V_X(x) | x \in X \} = \{ V_X(x) | x \in X, A \in V_X(x) \} = \{ V_X(x) | x \in A \}$.

Let us now suppose that $\{ V_X(x) | x \in A \} \cap cl_{X^S}(\{ A \}) = \emptyset$, this means that $\forall x \in A : \exists Y \in V_X'(V_X(x)) : A \notin Y$, where $Y = \{ B O$-stack $| V' \in B \}$ for some open $V' \in V_X'(x)$. Therefore we have $\forall x \in A : \exists V_x = V' \in V_X'(x) : V_x \notin A$. Considering $\cup_{x \in A} V_x \supset A$, and the fact that $A$ is a stack we get $\cup_{x \in A} V_x \in A$. Since $A$ is also an $O$-stack all an $V_x$ were chosen to be open, we have that there is at least one $V_x$ in $A$, which contradicts our choice of $V_x$, hence $\{ V_X(x) | x \in A \} \cap cl_{X^S}(\{ A \})$ can not be empty, so $A \in b_{X^S}(r^S(X))$.  

We shall now show that the $CCl_0$-spaces form the firm $E_{Cl_0}$-reflective subcategory of $Cl_0$, hence they can be considered as the complete objects of $Cl_0$.

Proposition 4.1.21. $S_2$ is an $CCl_0$-object, hence for the epireflective hulls in $Cl_0$ we have: $E_{Cl_0}(\{ S_2 \}) \subset E_{Cl_0}(CCl_0) = CCl_0$.

Proposition 4.1.22. Any $CCl_0$-object is a $b$-closed subspace of a power of $S_2$. 

---
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Proof. Let \( X \) be an \( \text{CCl}_0 \)-object, since it is \( T_0 \) it is embedded in a power \( Y = S_2^J \).
We shall prove that \( X \) is \( b \)-closed in \( Y \). Let \( a \in Y \setminus X \). If \( cl_Y(\{a\}) \cap X = \emptyset \) we have \( a \not\in b_Y(\{X\}) \). If \( cl_Y(\{a\}) \cap X \neq \emptyset \) then there is a \( z \in X \) such that \( cl_X(\{z\}) = cl_Y(\{a\}) \cap X \). Since \( Y \) is \( T_0 \) and \( z \in cl_Y(\{a\}) \) it cannot be that \( a \in cl_Y(\{z\}) \), because then \( a \) would be \( z \). Therefore \( a \in Y \setminus cl_Y(\{z\}) = A \) and \( A \cap cl_Y(\{a\}) \cap X = \emptyset \). We conclude that \( a \not\in b_Y(X) \).

Proposition 4.1.23. \( \text{CCl}_0 = \mathcal{E}_{\text{Cl}_0}(\{\subseteq\}) \)

Proof. We already have one inclusion. Because \( \text{Cl}_0 \) is co-well-powered ([20]), \( \mathcal{E}_{\text{Cl}_0}(\{\subseteq\}) \) consists of all \( b \)-closed subspaces of powers of \( S_2 \). Hence by the previous proposition any \( \text{CCl}_0 \)-object must be in \( \mathcal{E}_{\text{Cl}_0}(\{\subseteq\}) \).

Example 4.1.24. [The \( \text{CCl}_0 \)-reflection the closure space from Example 1.2.22.]
Let us consider again the closure space \( X \) described in Example 1.2.22. We recall that \( X = \{a, b, c\} \) and that the open subsets are

\[ \mathcal{O}_X = \{\{a, b, c\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \emptyset, \{b, c\}\} \]

From Example 1.2.22 we know that this space is totally disconnected, hence it cannot be a point-closure space. We have seen that every \( T_0 \) closure space \( X \) is a subspace of \( S_2^C(X, \subseteq) \). In fact when we take a closer look at the proof of this proposition, we see that it suffices to consider the class of mappings \( C = \{I_A : X \rightarrow S_2 | A \text{ proper open set of } X\} \). Hence for our example, we write the proper open sets as \( A_1 = \{b\}, A_2 = \{c\}, A_3 = \{a, b\}, A_4 = \{a, c\}, A_5 = \{b, c\} \) and we get a natural embedding

\[ i : X \rightarrow S_2^5 : x \mapsto (I_{A_1}(x), I_{A_2}(x), I_{A_3}(x), I_{A_4}(x), I_{A_5}(x)) \]

With this, \( X \) is isomorphic to the subspace

\[ i(X) = \{(0,0,1,1,0),(1,0,1,0,1),(0,1,0,1,1)\} \]

of \( S_2^5 \). The closed sets of this subspace are given in Figure 4.1. Using the \( b \)-closure one finds that that the \( \text{CCl}_0 \)-reflection of \( X \) is the subspace

\[ \{(0,0,1,1,0),(1,0,1,0,1),(0,1,0,1,1),(1,1,1,1,1),(0,1,1,1,1),(1,0,1,1,1)\} \]

of \( S_2^5 \), with closed sets as in Figure 4.1. Remark that this space is isomorphic to the space one obtains from lattice of open sets of \( X \) (see cover) by means of the functor \( S \) described in Proposition 3.1.4 or to the space \( X_S \).

\[ \Box \]
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Figure 4.1: Example 1.2.22 as a subspace of $S^5_2$ and its $b$-closure.

4.1.3 $\mathbf{0Cl}_0$ has no firm $\mathcal{U}_{\mathbf{0Cl}_0}$-reflective subcategory

We write $\mathbf{0Cl}_0$ for the full subcategory of $\mathbf{Cl}$ given by all the zero-dimensional $T_0$ spaces.

**Proposition 4.1.25.** The zero-dimensional $T_0$ closure spaces are exactly the subspaces of powers of the discrete space $D_2$.

**Proof.** Clearly any subspace of a power of $D_2$ is zero-dimensional and $T_0$. Conversely let $\overline{X}$ be a zero-dimensional $T_0$ closure space. Consider the following source:

$$\langle f : \overline{X} \to D_2 \rangle_{f \in C(\overline{X}, D_2)}$$

here $C(\overline{X}, D_2)$ denotes the set of all continuous functions from $\overline{X}$ to $D_2$. This source is initial, since if $A$ is a clopen set of $\overline{X}$ we can consider the indicator $I_A : \overline{X} \to D_2$, which is continuous. Hence $A$ is clopen in the initial object $\overline{X}_{in}$ and $\text{CO}(\overline{X}) \subset \text{CO}(\overline{X}_{in})$. Conversely we clearly have $\text{CO}(\overline{X}_{in}) \subset \text{CO}(\overline{X})$. The above source is also point-separating since if $x \neq y$ there exists a clopen $V$ which separates $x$ and $y$, such that $x \in V$ and $y \notin V$, then $I_V(x) \neq I_V(y)$. We now have the following embedding:

$$i : \overline{X} \to D_2^{C(\overline{X}, D_2)} : x \mapsto \langle f(x) \rangle_{f \in C(\overline{X}, D_2)}$$

Hence $\overline{X}$ is a subspace of a power of $D_2$. \hfill $\Box$

Since the zero-dimensional closure spaces are the subspaces of powers of $D_2$, the regular $\mathbf{0Cl}_0$-closure on $\overline{X}$ can be characterized as:

$$x \in \text{reg}_{\mathbf{0Cl}_0}^{\overline{X}}(A) \iff \forall f, g : \overline{X} \to D_2 : f|_A = g|_A \Rightarrow f(x) = g(x)$$
where $A \subset X$. We also define the following closure:

$$x \in \eta_X(A) \iff \exists \text{ clopen } V, U \in \mathcal{V}_X(x) : V \cap A = (X \setminus U) \cap A$$

**Proposition 4.1.26.** $reg^{0Cl}_0 = \eta$

**Proof.** Let $X$ be a zero-dimensional Hausdorff closure space and $A$ a subset of $X$. If $x \notin reg^{0Cl}_0(A)$ there are $f, g : X \to D_2$ such that $f|_A = g|_A$ and $f(x) \neq g(x)$, i.e. $f(x) = 1, g(x) = 0$. Thus we have clopen neighborhoods $f^{-1}(1)$ and $g^{-1}(0)$ of $x$ for which $f^{-1}(1) \cap A = (X \setminus g^{-1}(0)) \cap A$ hence $x \notin \eta_X(A)$. Conversely if $x \notin \eta_X(A)$ we have two clopen neighborhoods $U$ and $V$ of $x$ for which $V \cap A = (X \setminus U) \cap A$. The morphisms $I_V$ and $I_{(X \setminus U)}$ coincide on $A$ but differ in $x$, hence $x \notin reg^{0Cl}_0(A)$. \[ \square \]

In order to prove that $0Cl_0$ has no firm $U_{0Cl_0}$-reflective subcategory we introduce the following spaces:

$$\mathbb{N} = D_\mathbb{N}$$

$$\mathbb{N}^\infty = (\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}, \mathcal{O}_{\mathbb{N}^\infty})$$

where $\mathcal{O}_{\mathbb{N}^\infty} = \{ A \subset \mathbb{N} \cup \{\infty\} | \infty \in A \Rightarrow \mathbb{N} \setminus A \text{ is finite} \}$ are the open sets of $\mathbb{N}^\infty$. These spaces are zero-dimensional and $T_0$. Moreover $\mathbb{N}$ is a subspace of $\mathbb{N}^\infty$ and we have an embedding $i : \mathbb{N} \to \mathbb{N}^\infty$.

**Proposition 4.1.27.** $i : \mathbb{N} \to \mathbb{N}^\infty$ is an epimorphism in $0Cl_0$.

**Proof.** consider the point $\infty$ and suppose $\infty \notin \eta_{\mathbb{N}^\infty}(\mathbb{N})$. Then there are clopen neighborhoods $U, V$ of $\infty$ such that $V \cap N = (\mathbb{N}^\infty \setminus U) \cap N$. Since $\infty \in V$ and $\infty \in U$ the left hand side is infinite while the right hand side is finite. Hence there are no such $U, V$ and thus $i$ is $\eta$-dense. \[ \square \]

**Proposition 4.1.28.** $D_2$ is not $U_{0Cl_0}$-injective.

**Proof.** Consider the following morphisms:

$$\begin{array}{ccc}
\mathbb{N} & \xrightarrow{i} & \mathbb{N}^\infty \\
\downarrow f & & \\
D_2 & \xrightarrow{} &
\end{array}$$

where $f : \mathbb{N} \to D_2 : n \mapsto \begin{cases} 1 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$. $f$ is a morphism since $\mathbb{N}$ is a discrete space. However if there was an extension $\bar{f} : \mathbb{N}^\infty \to D_2$ then $\bar{f}^{-1}(0)$ and $\bar{f}^{-1}(1)$
would be clopen in $N^\infty$. If $\infty \in \bar{f}^{-1}(0)$ then $f^{-1}(1)$ would be finite which is impossible, however for the same reasons $\infty \in \bar{f}^{-1}(1)$ is also impossible. Hence there can be no such extension.

Observing the fact that any zero-dimensional $T_0$ space with more than one point, contains a subspace isomorphic to $D_2$ we can conclude that the only $U_0\text{Cl}_0$-injective objects are one-point spaces. Since these spaces do not form a $U_0\text{Cl}_0$-reflective subcategory of $0\text{Cl}_0$, the latter has no firm $U_0\text{Cl}_0$-reflective subcategory.

Later on we will introduce non-Archimedean spaces in the context of uniformizable closure spaces as a uniform counterpart of zero-dimensional closure spaces. For this category of spaces we will be able to give a characterization of the complete objects.

### 4.2 Firmness in coreflective hereditary subcategories

We will now consider a coreflective hereditary subcategory $C$ of some well-fibred topological construct $X$ and look at the firm $U_{T_0C}$-reflective subcategory of $T_0C$. Afterwards we will give some examples in $\text{Top}$ and $\text{Cl}$.

#### 4.2.1 The firm $U_{T_0C}$-reflective subcategory of $T_0C$

In this section subcategories will always be full and iso-closed. We will work in a well-fibred topological construct $X$ and as before the indiscrete object on $\{0,1\}$ will be written as $I_2$.

**Remark 4.2.1.** [34],[37],[43] Let $X$ be a well-fibred topological construct. $T_0X$ is an extremally epireflective subcategory, hence complete, well-powered and closed under formation of initial mono-sources in $X$. ◄

**Corollary 4.2.2.** Let $X$ be a well-fibred topological construct, then $\text{Emb } T_0X = \text{Emb } X \cap \text{Mor } T_0X$.

We now consider a coreflective subcategory $C$ of a well-fibred topological construct $X$ with underlying functor $U$. We will write $R^c$ for the coreflection
functor. For any object $X \in |X|$ we’ll write the reflection morphism as $c_X : X \to R^c X$ where $U(c_X) = 1_{c_X}$. We know that $C$, with the restriction of $U$, forms a well-fibred topological construct. Moreover the $C$-initial object which lifts the structured source $(f_i : X \to Y_i)_{i \in I}$, $Y_i \in |C|$ is given by the coreflection $R^c X_{in}$ of the initial $X$-object lifting the same source. Since $C$ is a well-fibred topological construct, it is complete and well-powered. Moreover the previous remark applies to $C$.

**Proposition 4.2.3.** Let $X$ be a well-fibred topological construct and let $C$ be a coreflective subcategory of $X$. Then $Y \in |T_0 C| \Rightarrow Y \in |T_0 X|$

*Proof.* Let $Y$ be a $T_0 C$-object. The indiscrete two-point object of $C$ is $R^c L_2$, i.e. the coreflection of $L_2$. Hence for any $X$-morphism $g : L_2 \to Y$ we have the following diagram:

$$
\begin{array}{ccc}
L_2 & \xrightarrow{g} & Y \\
|c_{L_2}| & \downarrow & \\
R^c L_2 & \xrightarrow{g \circ |c_{L_2}|} & \\
\end{array}
$$

Since $Y$ is a $T_0$-object of $C$ we have that $g \circ |c_{L_2}|$ must be a constant morphism. However $U(|c_{L_2}|)$ is the identity on $\{0, 1\}$, so $g$ is constant. Finally, this allows us to conclude that $Y \in T_0 X$. 

We will now consider the full subcategory $T_0 X \cap C$.

**Proposition 4.2.4.** Let $X$ be a well-fibred topological construct and let $C$ be a coreflective subcategory of $X$. $T_0 X \cap C$ is extremally epireflective in $C$.

*Proof.* Let $X \in |C|$. We construct the epireflection as follows. Writing $r_X : X \to R_X$ for the $T_0 X$-epireflection of an $C$-object and the $C$-coreflection $c_X : X \to R^c X$, we consider the unique factorization $r_{X}^c$ given by diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{r_X} & R_X \\
|c_X| & \downarrow & \\
R^c X & \xrightarrow{c_X} & \\
\end{array}
$$

Since $r_X = c_X \circ r_X^c$ and $r_X$ is an extremal epimorphism we have that $c_X$ is an isomorphism. Therefore $r_{X}^c = c_X^{-1} \circ r_X^c$ is an extremal epimorphism. The universal property follows immediately. 


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**Corollary 4.2.5.** Let $X$ be a well-fibred topological construct and let $C$ be a coreflective subcategory of $X$. Then

$$T_0C = T_0X \cap C$$

**Proof.** The subcategory $T_0X \cap C$ is epireflective but not bireflective in $C$ (if it were then $T_0X$ would be bireflective in $X$). Moreover this subcategory contains $T_0C$ by Proposition 4.2.3. Since $T_0C$ is the largest epireflective subcategory of $C$, which is not bireflective we have $T_0C = T_0X \cap C$. \qed

**Proposition 4.2.6.** Let $X$ be a well-fibred topological construct and let $C$ be a coreflective subcategory of $X$. We have:

$$X \in |T_0X| \Rightarrow R^cX \in |T_0C|$$

**Proof.** Let $X \in |T_0X|$. Consider an $X$-morphism $f : I \to R^cX$ and the coreflection $c_X : R^cX \to X$. Obviously $c_X \circ f$ must be constant since $X \in |T_0X|$. Hence $f$ is also constant. \qed

**Corollary 4.2.7.** Let $X$ be a well-fibred topological construct. For any coreflective subcategory $C$ of $X$, $T_0C$ is coreflective in $T_0X$.

We summarize the above results in the following scheme.

```
    X
     \^ r
      \  \\
       \ T_0X
          \ c
           \  \\
            \  C
               \ c
                \  \\
                 \ T_0C
```

**Proposition 4.2.8.** Let $X$ be a well-fibred topological construct. If $C$ is a hereditary coreflective subcategory of $X$, then $\text{Emb } T_0C = \text{Emb } T_0X \cap \text{Mor } C$.

**Proof.** Suppose that $m : X \to Y$ is an embedding in $T_0C$, as such it is also an embedding in $C$ (Corollary 4.2.2). Since it is an initial $C$-morphism, it is...
obtained by means of the coreflection and an initial $X$-morphism $m^* : X_{in} \to Y$:

$$
\begin{array}{c}
X_{in} \\
\downarrow c_{X_{in}} \\
X = \text{Reg} X_{in}
\end{array}
\xrightarrow{m^*} 
\begin{array}{c}
Y \\
\downarrow c_Y
\end{array}
$$

because $m^* : X_{in} \to Y$ is an embedding in $X$ and $Y \in |C|$, it follows from the hereditariness of $C$ that $X_{in} \in |C|$. Hence $X = \text{Reg} X_{in} = X_{in}$ thus $m : X \to Y$ is an embedding in $X$, between $T_0$-objects, therefore it is an embedding in $T_0X$.

Conversely, let $m : \underline{X} \to \underline{Y}$ be an embedding between $C$-objects in $T_0X$. By Corollary 4.2.2 it is also an embedding in $X$. Since both $\underline{X}$ and $\underline{Y}$ are in $|T_0X \cap C| = |T_0C|$ we finally have that $m : \underline{X} \to \underline{Y}$ is an embedding in $T_0C$. \(\Box\)

Let $X$ and $C$ be as above ($C$ not necessarily hereditary). Suppose there is a class $P$ of $C$-objects in $T_0X$ which $\text{Emb} T_0X$-cogenerates $T_0X$:

$$
\begin{array}{c}
X \\
\downarrow c \\
C
\end{array}
\xleftarrow{\text{Emb } T_0X \text{-cogen.}} 
\begin{array}{c}
T_0X \\
\downarrow c \\
T_0C
\end{array}
\xleftarrow{\text{P}}
$$

We have the following proposition.

**Proposition 4.2.9.** Let $X$ be a well-fibred topological construct and let $C$ be a coreflective subcategory of $X$. Suppose there is a class $P$ of $C$-objects in $T_0X$ which $\text{Emb} T_0X$-cogenerates $T_0X$. Then $\text{Reg}_{T_0X} = \text{Reg}_{T_0C}$ for each $T_0$-$C$-object $X$.

**Proof.** Because $T_0C$ is a subcategory of $T_0X$ we know from the definition of the regular closure that $\text{Reg}_{T_0X}(M) \subset \text{Reg}_{T_0C}(M)$, for any $X \in |T_0C|$. On the other
hand, we know that \( \mathbf{P} \ Emb \ T_0 \mathbf{X} \)-cogenerates \( T_0 \mathbf{X} \), hence for \( \mathbf{X} \in \vert T_0 \mathbf{C} \vert \) and \( M \subset X \) we have that whenever \( x \notin \text{reg}_{\mathbf{X}}(M) \) there exist two \( T_0 \mathbf{X} \)-morphisms \( f, g : \mathbf{X} \to P \) (\( P \in \mathbf{P} \)) such that \( f_{\vert M} = g_{\vert M} \) and \( f(x) \neq g(x) \). Since \( \mathbf{X} \) and \( P \) are \( T_0 \mathbf{C} \)-objects, \( f \) and \( g \) are \( T_0 \mathbf{C} \)-morphisms, hence \( x \notin \text{reg}_{\mathbf{X}}(M) \). Thus we have \( \text{reg}_{\mathbf{X}}(M) = \text{reg}_{\mathbf{X}}(M) \) whenever \( \mathbf{X} \in \vert T_0 \mathbf{C} \vert \).

**Corollary 4.2.10.** Let \( \mathbf{X} \) be a well-fibred topological construct and let \( \mathbf{C} \) be a coreflective subcategory of \( \mathbf{X} \). Suppose there is a class \( \mathbf{P} \) of \( \mathbf{C} \)-objects in \( T_0 \mathbf{X} \) which \( \text{Emb} \ T_0 \mathbf{X} \)-cogenerates \( T_0 \mathbf{X} \). Then \( \text{Epi} \ T_0 \mathbf{C} = \text{Epi} \ T_0 \mathbf{X} \cap \text{Mor} \ T_0 \mathbf{C} \).

Combining Proposition 4.2.8 and Corollary 4.2.10 we have another corollary.

**Corollary 4.2.11.** Let \( \mathbf{X} \) be a well-fibred topological construct and let \( \mathbf{C} \) be a hereditary coreflective subcategory of \( \mathbf{X} \). Suppose there is a class \( \mathbf{P} \) of \( \mathbf{C} \)-objects in \( T_0 \mathbf{X} \) which \( \text{Emb} \ T_0 \mathbf{X} \)-cogenerates \( T_0 \mathbf{X} \). For the class \( \mathcal{U}_{T_0 \mathbf{C}} \) of epimorphic embeddings of \( T_0 \mathbf{C} \), the following holds:

\[
\mathcal{U}_{T_0 \mathbf{C}} = \mathcal{U}_{T_0 \mathbf{X}} \cap \text{Mor} \ T_0 \mathbf{C}
\]

**Proposition 4.2.12.** Let \( \mathbf{X} \) be a well-fibred topological construct and \( \mathbf{C} \) a hereditary coreflective subcategory of \( \mathbf{X} \). Suppose there is a class \( \mathbf{P} \) of \( \mathbf{C} \)-objects in \( T_0 \mathbf{X} \) which \( \text{Emb} \ T_0 \mathbf{X} \)-cogenerates \( T_0 \mathbf{X} \). Every \( \mathcal{U}_{T_0 \mathbf{X}} \)-injective object in \( T_0 \mathbf{C} \) is \( \mathcal{U}_{T_0 \mathbf{C}} \)-injective, i.e. \( \text{Inj}(\mathcal{U}_{T_0 \mathbf{X}}) \cap T_0 \mathbf{C} \subset \text{Inj}(\mathcal{U}_{T_0 \mathbf{C}}) \).

**Proof.** Let \( Z \) be an \( \mathcal{U}_{T_0 \mathbf{X}} \)-injective object of \( T_0 \mathbf{C} \). Take \( u : X \to Y \in \mathcal{U}_{T_0 \mathbf{C}} \) and \( f : X \to Z \). By Corollary 4.2.11 \( u \in \mathcal{U}_{T_0 \mathbf{X}} \). Since \( Z \) is in \( \text{Inj}(\mathcal{U}_{T_0 \mathbf{X}}) \) we have an extension \( f^* \) of \( f \) which makes the following diagram commute.

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{f^*} \\
Z & & \\
\end{array}
\]

Since both \( Z \) and \( Y \) are \( T_0 \)-objects in \( \mathbf{C} \), \( f^* \) is a \( T_0 \mathbf{C} \)-morphism. Thus \( Z \) is \( \mathcal{U}_{T_0 \mathbf{C}} \)-injective.

**Proposition 4.2.13.** Let \( \mathbf{X} \) be a well-fibred topological construct. Let \( \mathbf{C} \) be a coreflective subcategory of \( \mathbf{X} \) and suppose there is a class \( \mathbf{P} \) of \( \mathbf{C} \)-objects in \( T_0 \mathbf{X} \) which \( \text{Emb} \ T_0 \mathbf{X} \)-cogenerates \( T_0 \mathbf{X} \). In this case \( T_0 \mathbf{C} \) is \( \text{Emb} \ T_0 \mathbf{C} \)-cogenerated by \( \mathbf{P} \).

**Proof.** Let \( X \in \vert T_0 \mathbf{C} \vert \) then \( X \) is also a \( T_0 \)-object in \( \mathbf{X} \), hence there is an embedding \( m : X \to \Pi_{i \in I} P_i \) in \( \mathbf{X} \). So by Remark 4.2.1 \( R^{\ast}(m) : X \to R^{\ast} \Pi_{i \in I} P_i \) is also an embedding in \( T_0 \mathbf{C} \). Hence \( T_0 \mathbf{C} \) is \( \text{Emb} \ T_0 \mathbf{C} \)-cogenerated by \( \mathbf{P} \). \[\square\]
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By Remark 4.2.1 the initially structured categories $T_0\mathbf{X}$ and $T_0\mathbf{C}$ are complete and well-powered, hence we can apply Theorem 4.1.6.

**Theorem 4.2.14.** Let $\mathbf{X}$ be a well-fibred topological construct and $\mathbf{C}$ a hereditary coreflective subcategory of $\mathbf{X}$. Suppose $\mathbf{P}$ is a class of $\mathcal{U}_{T_0\mathbf{X}}$-injective $\mathbf{C}$-objects in $T_0\mathbf{X}$, which $\text{Emb } T_0\mathbf{X}$-cogenerates $T_0\mathbf{X}$. If $\mathbf{B}$ is one of $T_0\mathbf{X}$ or $T_0\mathbf{C}$, then $\mathbf{B}$ admits a unique firmly $\mathcal{U}_B$-reflective subcategory $\mathbf{C}_B$. Moreover

$$CB = \text{Inj}(\mathcal{U}_B) = \mathcal{E}_B(\mathbf{P})$$

We schematize the previous results as follows.

We will now characterize these complete objects as certain subspaces of products of $\mathbf{P}$-objects.

**Proposition 4.2.15.** Let $\mathbf{C}$ be a coreflective subcategory of a well-fibred topological construct $\mathbf{X}$ and suppose there is a class $\mathbf{P}$ of $\mathbf{C}$-objects in $T_0\mathbf{X}$ which $\text{Emb } T_0\mathbf{X}$-cogenerates $T_0\mathbf{X}$. If $T_0\mathbf{X}$ is co-well-powered then $T_0\mathbf{C}$ is also co-well-powered.

**Proof.** Let $\overline{X}$ be a fixed $T_0\mathbf{C}$ object and let $\mathcal{E}$ be a set of representatives for the epimorphisms $e : \overline{X} \to Y$ in $T_0\mathbf{X}$. Write $\mathcal{E}'$ for $\mathcal{E} \cap \text{Mor } T_0\mathbf{C}$. If $f : \overline{X} \to Y$ is an epimorphism of $T_0\mathbf{C}$ then by Corollary 4.2.10 it is also an epimorphism in $T_0\mathbf{X}$, hence there is an $f' : \overline{X} \to Z$ in $\mathcal{E}$ and an isomorphism $h_f : Z \to Y$ such that $f = h_f \circ f'$. From this it follows $Z \in |T_0\mathbf{C}|$, hence $f' \in \mathcal{E}'$ and thus $T_0\mathbf{C}$ is co-well-powered. \qed
Proposition 4.2.16. Let $X$ be a well-fibred topological construct and let $B$ be an extremally epireflective subcategory of $X$. If $\text{reg}^B$ is weakly hereditary then the extremal monomorphisms of $B$ are exactly the $\text{reg}^B$-closed embeddings of $B$.

Proof. Because $B$ is an extremally epireflective subcategory of a topological category, we know by Proposition 4.1.11 that every regular-closed embedding of $B$ is an extremal monomorphism. Since $B$ is initially structured every extremal monomorphism is an embedding in $B$ [37]. Suppose $m : X \to Y$ is an extremal monomorphism in $B$, we consider the following canonical factorization.

\[
\begin{array}{ccc}
X & \xrightarrow{m} & Y \\
\downarrow{e} & & \downarrow{m^*} \\
\text{reg}^B_Y(X) & & \\
\end{array}
\]

Since $\text{reg}^B$ is weakly hereditary $e$ is an epimorphism, hence an isomorphism. So $m$ is a $\text{reg}^B$-closed embedding of $B$. \qed

Since $B$ is complete and well-powered it follows from [43] that when $B$ is co-well-powered the epireflective hulls in $B$ are given by

\[
\mathcal{E}_B(P) = \{ X \in |B| \mid \exists m : X \to \Pi_{i \in I} P_i \text{ extremal monomorphism in } B, P_i \in P \}
\]

here the products are also taken in $B$. With this characterization of $\mathcal{E}_B(P)$ we can reformulate Theorem 4.2.14 as follows.

Theorem 4.2.17. Let $X$ be a well-fibred topological construct and $C$ a hereditary coreflective subcategory of $X$. Suppose $P$ is a class of $\mathcal{U}_{T_0X}$-injective $C$-objects in $T_0X$, which $\text{Emb} T_0X$-cogenerates $T_0X$. If $B$ is one of $T_0X$ or $T_0C$ and when $T_0X$ is co-well-powered and $\text{reg} T_0X$ is weakly hereditary the subcategory $CB$ consists of the regular-closed subspaces of products of $P$-spaces, taken in $B$.

4.2.2 Situation for $\text{Top}_0$ and $\text{Cl}_0$

Let $\text{Cl}$ be the category of closure spaces and continuous maps. Let $\text{Top}$ be the full subcategory of topological spaces. We know that $\text{Top}$ is a coreflective hereditary subcategory of $\text{Cl}$. From our earlier results we know that the category $\text{Cl}_0$ of $T_0$-objects of $\text{Cl}$ is $\text{Emb} \text{Cl}_0$-cogenerated by the Sierpinski space $\Sigma_2$, which is $\mathcal{U}_{\text{Cl}_0}$-injective, and that the regular closure induced by $\text{Cl}_0$ is the $b$-closure. Moreover $\text{Cl}_0$ is co-well-powered, thus applying Theorem 4.2.17 we get the following.
4. Completeness

**Theorem 4.2.18.** Consider $\mathcal{C}l_0$ and $\mathsf{Top}_0$. $\mathcal{C}l_0$ has a unique firm $\mathcal{U}_{\mathcal{C}l_0}$-reflective subcategory and $\mathsf{Top}_0$ has a unique firm $\mathcal{U}_{\mathsf{Top}_0}$-reflective subcategory. These full subcategories are given by the objects which are injective w.r.t. the $b$-dense embeddings of $\mathcal{C}l_0$, resp. $\mathsf{Top}_0$. These are also the $b$-closed subspaces of powers (taken in $\mathcal{C}l_0$, resp. $\mathsf{Top}_0$) of $S_2$.

From our previous work we know this unique firm $\mathcal{U}_{\mathcal{C}l_0}$-reflective subcategory $\mathsf{CC}l_0$ of complete objects of $\mathcal{C}l_0$ consists of the point-closure spaces. From general topology it is known that the unique firm $\mathcal{U}_{\mathsf{Top}_0}$-reflective subcategory of $\mathsf{Top}_0$ is $\mathsf{Sob}$, the category of the sober topological spaces. Although $\mathsf{Sob}$ is quite different from $\mathsf{CC}l_0$ (cfr. Remark 3.1.12), the objects in these categories are formed in the same way: as $b$-closed subspaces of powers of $S_2$, only the category in which these powers are taken differs. This is summarized in the following scheme:

**Example 4.2.19.** [A sober topological space which as a closure space is not in $\mathsf{CC}l_0$.]

The discrete closure space $D_2$ on two points is a topological Hausdorff space, hence from general topology we know this is a sober topological space. However as a closure space it is clearly not a point-closure space, i.e. it is a $b$-closed subspace of a topological power of $S_2$ but it is not $b$-closed as a subspace of a power of $S_2$ in $\mathcal{C}l$. ▸
4. Completeness

4.2.3 Classes of coreflective hereditary subcategories of Cl and Top: Tight($\alpha$)-spaces

Definition 4.2.20. Let $\alpha$ be an infinite cardinal. We define the following full subcategory of Cl given by the objects:

$$\text{Tight}(\alpha) = \{ X \in |\text{Cl}| | x \in cl_X M \Rightarrow \exists M' \subset M, |M'| < \alpha : x \in cl_X M' \}$$

Proposition 4.2.21. Tight($\alpha$) is hereditary.

Proof. Let $X \in \text{Tight}(\alpha)$ and $M \subset Y \subset X$. Then $x \in cl_Y (M) \subset cl_X (M)$. So there is a $M' \subset M$ with $|M'| < \alpha$ such that $x \in cl_X (M')$ hence also $x \in cl_Y (M')$.

Example 4.2.22. [A Tight($\alpha^+$)-space $X$ such that $X^X$ is not in Tight($\alpha^+$)]

Let $X$ be any set with $|X| = \alpha$. We provide a closure structure on $X$ by the following base of open sets $\{ \{x, y\}| x, y \in X\}$, thus we get the closure space $\overline{X}$. Note that this space is a $T_0$ closure space (remark that this is not a topological space). Moreover we have $cl_{\overline{X}} M = X$ whenever $M = X \setminus \{x_0\}$ for some $x_0$ or when $M = X$, in all other cases $cl_{\overline{X}} M = M$. If we denote by $\alpha^+$ the successor of $\alpha$ we obviously have $X \in \text{Tight}(\alpha^+)$. Next we consider the product space $\overline{X}^X$ (which is $T_0$). Let $M$ be the subset $\Pi_{x_0 \in X}(X \setminus \{x_0\})$ of $X^X$ and $\bar{x} = (x_0)_{x_0 \in X}$. A base for the open subsets of $\overline{X}^X$ is given by the sets $pr_{\bar{x}}^{-1} (\{y, z\})$, a base for the neighborhoods of $\bar{x}$ is given by the sets $pr_{x_0}^{-1} (\{x_0, y\})$, which always have a nonempty intersection with $M$, hence $\bar{x} \in cl_{\overline{X}^X} M$.

Let us suppose that $\overline{X}^X$ is in Tight($\alpha^+$) then there is a $M' \subset M$ with $|M'| < \alpha^+$ such that $\bar{x} \in cl_{\overline{X}^X} M'$. If $M \neq M'$ then there exists a $\bar{z} \in M$ with $\bar{z} \notin M'$. For this $\bar{z}$ there exists an $x_0$ such that $y = \bar{z}_{x_0} \in pr_{x_0}(M)$ and $y \notin pr_{x_0}(M')$. Since $\{x_0, y\}$ is an open neighborhood of $x_0$ in $X$ for which $\{x_0, y\} \cap pr_{x_0}(M') = \emptyset$ we have $\bar{x} \notin cl_{\overline{X}^X} M'$, which contradicts the choice of $M'$. Hence $M' = M$, but then $|M'| = |M| = |\Pi_{x_0 \in X}(X \setminus \{x_0\})| = |(X \setminus \{x_0\})|^{|X|} = |X|^{|X|} = 2^\alpha \neq \alpha^+$, which is impossible. Therefore we know that $\overline{X}^X \notin \text{Tight}(\alpha^+)$. 

Corollary 4.2.23. Tight($\alpha$) is not productive, hence it can not be epireflective in Cl.
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Obviously one has that $\text{Tight}(\alpha) \subset \text{Tight}(\alpha^+)$ Moreover every closure space $X$, with $|X| = \alpha$ is in $\text{Tight}(\alpha^+)$, thus $\bigcup_{\alpha \in \text{CARD}} \text{Tight}(\alpha) = |\text{Cl}|$. So we get the following scheme.

A point-closure space is not necessarily a $\text{Tight}(\alpha)$-space as is shown in the next example.

**Example 4.2.24.** [A point-closure space which is not in $\text{Tight}(\omega)$]

If we consider the closure structure on the set $Y = \mathbb{N} \cup \{\infty\}$ given by the basis for closed sets $\{[0, n]| n \in \mathbb{N}\}$ (see Figure 4.2). This space $Y$ is clearly a $\text{CCl}_0$-space since $[0, n] = \text{cl}_Y \{n-1\}$ for every $n \in \mathbb{N}_0$ and $\mathbb{N} \cup \{\infty\} = \text{cl}_Y \{\infty\}$. But if $M = \mathbb{N}$ then $\mathbb{N} \cup \{\infty\} = \text{cl}_Y M$ and there is no finite subset $M'$ of $M$ such that its closure contains $\infty$. Hence it is not in $\text{Tight}(\omega)$.

![Figure 4.2: Closed sets of the space $Y$ from Example 4.2.24.](image)
We now introduce the following collections of spaces:

\[ G^\alpha = \{ X \in |\text{Cl}| \mid |X| < \alpha \} \]

\[ G_{gen}^\alpha = \{ X \in |\text{Cl}| \mid U \subset X \text{ closed in } X \iff \forall A \subset X, |A| < \alpha : U \cap A \text{ closed in } A \} \]

**Proposition 4.2.25.** Let \( f : X \rightarrow Y \) be a map with domain \( X \in G_{gen}^\alpha \), then \( f \) is continuous if and only if \( \forall A \subset X, |A| < \alpha : f|_A : A \rightarrow Y \) is continuous.

**Proof.** One implication is trivial, for the other we consider a closed set \( U \subset Y \). Then \( f|_A^{-1}(U) \) is closed in \( A \) for any \( A \subset X, |A| < \alpha \) hence \( f^{-1}(U) \cap A \) is closed in \( A \). Since \( X \) is in \( G_{gen}^\alpha \) we have that \( f^{-1}(U) \) is closed in \( X \). \( \square \)

**Proposition 4.2.26.**

\[ X \in G_{gen}^\alpha \iff X \text{ final for the sink } (j_A : A \rightarrow X)_{A \subset X, |A| < \alpha} \]

If \( X \) is a closure space we define a new closure space \( R^\alpha X \) on \( X \) by saying that a set \( U \subset X \) is closed in \( R^\alpha X \) if and only if \( \forall B \subset X, |B| < \alpha : U \cap B \text{ closed in } B \)

**Proposition 4.2.27.** The following hold:

1. Every closed set of \( X \) is a closed set of \( R^\alpha X \).
2. If \( B \subset X \) and \( |B| < \alpha \) then the closure structure induced on \( B \) by \( X \) and the one induced by \( R^\alpha X \) coincide.

**Proof.** We only prove the second part, since the first is trivial. Let \( B \subset X \), \( |B| < \alpha \) and let \( U \) be a closed subset of \( B \) induced by \( R^\alpha X \). There is a \( U' \) which is closed in \( R^\alpha X \) and such that \( U = U' \cap B \). Since \( |B| < \alpha \) we know that \( U = U' \cap B \) is closed in the subspace \( B \) induced by \( X \). \( \square \)

**Proposition 4.2.28.** \( G_{gen}^\alpha \) is coreflective in \( \text{Cl} \).

**Proof.** Let \( X \in |\text{Cl}| \), \( Z \in G_{gen}^\alpha \) and \( f : Z \rightarrow X \) continuous. We consider the following diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & X \\
\downarrow & \searrow & \downarrow 1_X \\
R^\alpha X \\
\end{array}
\]
4. Completeness

If \( A \subset Z, |A| < \alpha \) then of course \(|f(A)| < \alpha\), hence the closed subsets of \( B = f(A) \) are the same whether we consider it as a subspace of \( X \) or as a subspace of \( R^\alpha X \). Therefore \( f^* : A \to B \) is always continuous, thus \( f^* : Z \to R^\alpha X \) is continuous.

**Theorem 4.2.29.** For a closure space \( X \) the following are equivalent:

1. \( X \) is in the coreflective hull of \( G^\alpha \) in \( Cl \).
2. \( X \) is final for the sink \( (j_A : A \to X)_{A \subset X, |A| < \alpha} \).
3. \( X \) is a quotient of the coproduct of all its subspaces with cardinality less than \( \alpha \).
4. \( \forall B \subset X : \forall x \in X : x \in cl_X B \Leftrightarrow \exists Z \subset B, |Z| < \alpha : x \in cl_X Z. \)
5. \( \forall B \subset X : \forall x \in X : x \in cl_X B \Leftrightarrow \exists A \subset X, |A| < \alpha : x \in cl_X (A \cap B) \) and \( x \in A \).
6. \( U \subset X \) closed in \( X \Leftrightarrow \forall A \subset X, |A| < \alpha : U \cap A \) closed in \( A \).

**Proof.** We prove the following implications:

1 \( \Rightarrow \) 2: If \( X \) is in the coreflective hull of \( G^\alpha \) then it clearly is in \( G^\alpha_{gen} = \{ X \in [Cl][X] final for (j_A : A \to X)_{A \subset X, |A| < \alpha} \}. \)

2 \( \Rightarrow \) 3: This is obvious.

3 \( \Rightarrow \) 4: Suppose that \( \varphi : Y = \Pi_{A \subset X, |A| < \alpha} A \to X \) is final and surjective. Let \( B \subset X \) and \( x \in cl_X B \) then there exists a \( z \in \Pi_{A \subset X, |A| < \alpha} A \) for which \( \varphi(z) = x \) and there is a \( A \subset X \) with \(|A| < \alpha\) such that \( z \in cl_Y (\varphi^{-1}(B) \cap A) \). Let \( Z = \varphi^{-1}(B) \cap A \), since \( \varphi \) is continuous we have \( x = \varphi(z) \in \varphi(cl_Y (\varphi^{-1}(B) \cap A)) \subset cl_X (\varphi(\varphi^{-1}(B) \cap A)) \). Hence \( x \in cl_X Z \), \(|Z| < \alpha \) and \( Z \subset B \).

4 \( \Rightarrow \) 5: Let \( B \subset X \) and \( x \in cl_X B \), there is a \( Z \subset B \) with \(|Z| < \alpha\), such that \( x \in cl_X Z \). We choose \( A = Z \cup \{x\} \). Either \( x \in B \) and then \( x \in cl_X (B \cap A) \).

Or \( x \notin B \) so \( B \cap A = Z \), thus in either case \( x \in cl_X (B \cap A) \).

5 \( \Rightarrow \) 6: Let \( U \cap A \) be closed in \( A \) for any \( A \subset X, |A| < \alpha \) and let \( x \in cl_X U \).

There is such an \( A \) for which \( x \in cl_X (A \cap U) \) and \( x \in A \), so \( x \in cl_X (A \cap U) \cap A = cl_A (A \cap U) = A \cap U \). Hence \( x \in U \).
4. Completeness

6 ⇒ 1: If \( X \in G_{\alpha} \) it is final for \( (j_A : A \rightarrow X)_{A \subset X, |A| < \alpha} \), so \( X \) is in the coreflective hull of \( G^\alpha \).

If we write \( C(A) \) for the coreflective hull of a subcategory \( A \) of \( Cl \), we have the following corollary.

**Corollary 4.2.30.** \( C(G^\alpha) = G^\alpha = \text{Tight}(\alpha) \)

### 4.2.4 Firmly \( \mathcal{U} \)-reflective subcategories of \( \text{Tight}_0(\alpha) \)-spaces

We now have a collection \( \{ \text{Tight}(\alpha) | \alpha \in \text{CARD} \} \) of coreflective hereditary subcategories of the topological construct of closure space \( Cl \). By our observations in the previous section, we can apply Theorem 4.2.17.

Remark that the \( T_0 \)-objects of \( \text{Tight}(\alpha) \) are the \( T_0 \) closure spaces which are also \( \text{Tight}(\alpha) \)-objects, we shall write the full subcategory given by these spaces as \( \text{Tight}_0(\alpha) \) and the resulting \( T_0 \)-reflection of a \( \text{Tight}(\alpha) \)-space \( X \) as \( R_0^X \).

We know from Corollary 4.2.11 and in combination with the results from the previous section, that \( \mathcal{U}_{\text{Tight}_0(\alpha)} \) consists of all \( b \)-dense embeddings between \( \text{Tight}_0(\alpha) \)-spaces, to ease notation we shall simply write \( \mathcal{U}_\alpha \) for this class.

Applying Theorem 4.2.17 we get:

**Theorem 4.2.31.** There is a unique firm \( \mathcal{U}_\alpha \)-reflective subcategory \( CT\text{Tight}_0(\alpha) \) of \( \text{Tight}_0(\alpha) \), which coincides with the full subcategory \( \text{Inj}(\mathcal{U}_\alpha) \) of \( \mathcal{U}_\alpha \)-injective objects. It is given by the epireflective hull \( \mathcal{E}(\{S_2\}) \) in \( \text{Tight}_0(\alpha) \), these are the \( b \)-closed subspaces of powers of \( S_2 \), taken in \( CT\text{Tight}_0(\alpha) \).
Finally we have the following overview:

\[
\begin{align*}
\text{Cl} & \quad \text{CTight}_0 = \{S_2\} \\
\text{Tight}(\alpha) & \quad \text{Emb}\ Cl_0 \quad \text{cogen.} \\
\text{Emb}\ \text{Tight}_0(\alpha) & \quad \text{cogen.} \\
\{S_2\} & \\
\end{align*}
\]

The fact that \(\text{CCl}_0 = \text{CTight}_0(\alpha)\) does not (always) hold is given by the following proposition.

**Proposition 4.2.32.** Let \(\alpha\) be a successor cardinal. Then \(\text{CCl}_0 \neq \text{CTight}_0(\alpha)\).

**Proof.** First we observe that for any \(T_0\)-space \(X\) we have \(b_X(M) \subset \text{cl}_X(M)\). Since \(\alpha\) is a successor cardinal, and by Example 4.2.22 we can choose a \(T_0\) space \(X\) which is not in \(\text{Tight}_0(\alpha)\). Since it is \(T_0\) it can be embedded in some \(S_2^I\) (where the power is taken in \(\text{Cl}_0\)). By hereditariness of \(\text{Tight}_0(\alpha)\) we know that \(S_2^I \neq R_0^I S_2^I\). Hence there is a subspace \(A\) which is closed in \(R_0^I S_2^I\), but not in \(S_2^I\).

Since \(A\) is closed in \(R_0^I S_2^I\), it is also \(b\)-closed in this space, hence \(A\) is a \(\text{CTight}_0(\alpha)\)-space. We prove that \(A\) is not in \(\text{CCl}_0\). If it were a point-closure space, then there would be a unique \(a \in A\) for which \(A = \text{cl}_A(\{a\}) = \text{cl}_{R_0^I S_2^I}(\{a\}) \cap A\). We get that \(A \subset \text{cl}_{R_0^I S_2^I}(\{a\}) \subset \text{cl}_S^I(\{a\}) = \bigcap_{j \in J} \text{pr}_j^{-1}(\{0\})\), where \(J = \{j \in I | a_j = 0\}\) (\(a_j\) denotes \(\text{pr}_j(a)\)). Since \(A\) is not closed in \(S_2^I\), \(A \subset \bigcap_{j \in J} \text{pr}_j^{-1}(\{0\})\). Hence there is a \(b \notin A\) such that \(\forall j \in J : b_j = 0\). Consider \(A' = \{a, b\}\), clearly \(|A'| < \alpha\), hence \(A \cap A'\) is closed in \(A'\) as a subspace of \(S_2^I\). Thus \(\{a\} = A \cap A' = \bigcap_{j \in J} \text{pr}_j^{-1}(\{0\})\) and \(b \notin \bigcap_{j \in J} \text{pr}_j^{-1}(\{0\})\), this contradicts our choice of \(b\), hence \(A\) is not a \(\text{CCl}_0\)-space. \(\square\)

**Example 4.2.33.** [A closure space \(X\) having different completions in \(\text{Cl}_0\), in \(\text{Tight}_0(\beta)\) and in \(\text{Tight}_0(\omega)\), where \(\beta > 2^\omega\).]
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We construct a closure space $\mathcal{X}$ by taking $X = \mathbb{N}$ and closed sets $\mathcal{C}_X = \{\{0, n\} | n \in \mathbb{N}\} \cup \{\mathbb{N}\}$. As a $T_0$ space it can be embedded in $S_2^N$ as follows:

$$i : X \rightarrow S_2^N : x \mapsto (I_{[n, \infty]}(x))_{n \in \mathbb{N}}$$

The subspace $i(X)$ of $S_2^N$ contains all sequences of the form $\bar{z} = 1, 1, \ldots, 1, 0, 0, \ldots$.

The space $\mathcal{X}$ is in fact a $\text{Tight}_0(\omega)$-space, because if $x \in \text{cl}_X M = [0, n]$ then $n - 1 \in M$ and $x \in \text{cl}_X\{n - 1\}$. In the case where $x \in \text{cl}_X M = \mathbb{N}$ we have an $n \in M$ strictly greater than $x$ hence also $x \in \text{cl}_X\{n\}$.

We will now compare the completion of $\mathcal{X}$ in $\text{Cl}_0$, in $\text{Tight}_0(\beta)$ and in $\text{Tight}_0(\omega)$, where $\beta > 2^\omega$.

In order to find the $\beta$-closure of $i(X)$ in $S_2^N$ we consider three cases for a point $\bar{z} \not\in i(X)$:

1. $\bar{z}$ is not a constant sequence. Since $\bar{z} \not\in i(X)$, we have $k < l$ such that $\bar{z}_k = 0$ and $\bar{z}_l = 1$. Hence $\text{cl}_X\{\bar{z}\} \subset \text{pr}_k^{-1}(0)$ and $V = \text{pr}_l^{-1}(1)$ is an open neighborhood of $\bar{z}$. For those we have $V \cap \text{cl}_X\{\bar{z}\} \cap i(X) = \emptyset$, hence $\bar{z} \not\in \text{bcl}_i(X)$.

2. $\bar{z}$ is constant 0. In this case $\text{cl}_X\{\bar{z}\} = \{\bar{z}\}$ and since $\bar{z} \not\in i(X)$ we have $\bar{z} \not\in \text{bcl}_i(X)$.

3. $\bar{z}$ is constant 1. Thus $\text{cl}_X\{\bar{z}\} = S_2^N$ and because for any $i \in \mathbb{N}$: $\text{pr}_i^{-1}(1) \cap i(X) \neq \emptyset$, we know that this time $\bar{z} \in \text{bcl}_i(X)$.

In fact the subspace $\text{bcl}_i(X)$ of $S_2^N$ is homeomorphic with the space $\mathcal{Y}$ described in Example 4.2.24.

Since $|S_2^N| = 2^\omega < \beta$, $S_2^N$ is in $\text{Tight}_0(\beta)$. Hence the $\text{CTight}_0(\beta)$-reflection is the $\beta$-closure of $i(\mathbb{N})$ in $R_0^N S_2^N = S_2^N$, so it is the $\text{CCl}_0$-reflection $\mathcal{Y}$.

However, the space $\mathcal{Y}$ from Example 4.2.24 is not $\text{Tight}_0(\omega)$. Thus we know that $S_2^N$ cannot be a $\text{Tight}_0(\omega)$-space either. Therefore $R_0^N S_2^N \neq S_2^N$, hence the $\text{CTight}_0(\omega)$-reflection of $\mathcal{X}$ is the $\beta$-closure of $i(\mathbb{N})$ in $R_0^N S_2^N$, which can never be the same as the $\text{CTight}_0(2^\omega)$-reflection or $\text{CCl}_0$-reflection $\mathcal{Y}$, since the latter is not a $\text{Tight}_0(\omega)$-space.

However since $\text{cl}_{R_0^N S_2^N}\{\{\bar{z}\}\} \subset \text{cl}_{S_2^N}\{\{\bar{z}\}\}$ and any neighborhood of $\bar{z}$ in $S_2^N$ is also a neighborhood in $R_0^N S_2^N$ we have that $V \cap \text{cl}_{S_2^N}\{\{\bar{z}\}\} \cap i(X) = \emptyset$ implies
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\[ V \cap cl_{R_0\overline{\mathbb{S}^2}}(\{z\}) \cap i(X) = \emptyset \] so the \(b\)-closure of \(i(X)\) in \(R_0\overline{S^N}\) is a subset of its \(b\)-closure in \(\overline{S^N}\). Hence it is a Tight\(0\) subspace of \(Y\) which contains \(X\). Since both spaces differ only in one point and since \(Y\) is not a Tight\(0\) space, we can only conclude that \(\overline{X}\) is in fact itself a CTight\(0\) space. ▶

Since Tight\(\alpha\) and Top are both coreflective hereditary subcategories of Cl, we can construct the full subcategory Tight\(^t\)(\(\alpha\)) of topological Tight\(\alpha\)-spaces, which is a coreflective hereditary subcategory of Top. From general topology we know that Top\(_0\) is \(Emb\) Top\(_0\)-cogenerated by \(S_2\), which is \(U_{Top_0}\)-injective. Since Top\(_0\) is co-well-powered, we can apply Theorem 4.2.17 to the coreflective hereditary subcategories Tight\(^t\)(\(\alpha\)) of Top.

Again the \(T_0\)-objects of Tight\(^t\)(\(\alpha\)) are the \(T_0\) topological spaces of Tight\(^t\)(\(\alpha\)), hence we get the category Tight\(_0^t\)(\(\alpha\)). In the same manner as before we have that \(U_{Tight_0^t}(\alpha)\) consists of all \(b\)-dense embeddings between Tight\(_0^t\)(\(\alpha\))-spaces, to ease notation we shall simply write \(U_{\alpha}^t\) for this class.

Applying Theorem 4.2.17 we get:

**Theorem 4.2.34.** There is a unique firm \(U_{\alpha}^t\)-reflective subcategory CTight\(_0^t\)(\(\alpha\)) of Tight\(_0^t\)(\(\alpha\)), which which coincides with the full subcategory \(Inj(U_{\alpha}^t)\) of \(U_{\alpha}^t\)-injective objects. It is given by the epireflective hull \(E(\{S_2\})\) in Tight\(_0^t\)(\(\alpha\)), these are the \(b\)-closed subspaces of powers of \(S_2\), taken in CTight\(_0^t\)(\(\alpha\)).

Finally we have the following overview:
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These last results coincide with the ones that were already proven in [30], where $\text{Tight}^t(\alpha)$ was considered as an coreflective hereditary subcategory of the topological construct of affine sets $\mathbf{SSet}$.

**Example 4.2.35.** [The spaces $X$ and $Y$ from Example 4.2.33 considered in the topological setting.]

The spaces $X$ and $Y$ from Example 4.2.33, are in fact topological spaces. As such, the map $i$ can be used to embed $X$ in the space $S^{n_2}$, this time considered in $\text{Top}_0$. By a similar argument as in Example 4.2.33, one can show that the space $Y$ is again the $b$-closure of $X$. Thus in this case the sobrification yields a point-closure space. With the same arguments as before one shows that $Y$ is not a $\text{Tight}^t_0(\omega)$-space, so as a $\text{Tight}^t_0(\omega)$-space $X$ is complete, i.e. it is in $\text{CTight}^t_0(\omega)$.

\[\boxed{\text{92}}\]
Chapter 5

Uniform pre-nearness spaces

We have seen in earlier chapters that closure spaces arise naturally in different contexts (linear algebra, quantum logic, convex analysis, ...). Among these closure spaces the zero-dimensional ones play an important role. For example we have shown that zero-dimensional closures associated with a physical system represent the classical part of this system. Another example is the convex closure in a vector space. Throughout convex analysis the zero-dimensionality of convex is used to generate powerful theorems.

In the previous chapter we have also seen that the category $\textbf{0Cl}_0$ of zero-dimensional $T_0$ closure spaces, does not fulfill the expectations one might have when looking for complete objects. In the remaining part of this thesis we will introduce and study non-Archimedean spaces. This will be done in parallelism with the classical situation of topological spaces, where one first considers uniform spaces (or equivalently uniform nearness spaces) and then looks at those which have a base of equivalence relations (or partitions) called non-Archimedean uniform spaces [49], [50]. These non-Archimedean uniform spaces are the uniform counterparts of zero-dimensional topological spaces and they provide the right setting for a completion theory which incorporates the zero-dimensional topological spaces.

In this chapter we will develop a theory of uniform pre-nearness spaces and characterize these spaces using five different descriptions: uniform covers, near collections, small collections, entourages and pseudometrics. To obtain these
descriptions we will adapt the classical theories of uniform spaces (J.W. Tukey [45], N. Bourbaki [17]) and H. Herrlich’s theory of nearness spaces [31]. Our results will enable us to introduce non-Archimedean spaces in the next chapter, for which we will then formulate a suitable completion theory.

5.1 Preliminaries

5.1.1 Useful definitions and notations

In this section we recall and introduce some concepts and notations which we’ll be using throughout this chapter. For any set $X$ and $A \subset \mathcal{P}(X)$ we define the \text{sec} of $A$ as follows:

$$\text{sec} A = \{ B \subset X | \forall A \in A : A \cap B \neq \emptyset \}$$

We will also often make use of the \text{refinement} and \text{co-refinement} relations, for $A, B \in \mathcal{P}(X)$ we define:

$$A < B \iff \forall A \in A : \exists B \in B : A \subset B$$

$$A <_{co} B \iff \forall A \in A : \exists B \in B : B \subset A$$

With the above notations we introduce the corresponding stack and invert-stack operations.

$$\alpha \subset \mathcal{P}^2(X) :$$

$$\text{stack}_< \alpha = \{ B \subset \mathcal{P}(X) | \exists A \in \alpha : A < B \}$$

$$\text{stack}_{<_{co}} \alpha = \{ B \subset \mathcal{P}(X) | \exists A \in \alpha : A <_{co} B \}$$

$$\overline{\text{stack}}_{<_{co}} \alpha = \{ B \subset \mathcal{P}(X) | \exists A \in \alpha : B <_{co} A \}$$

For these operations we have the following rules:

**Lemma 5.1.1.** Let $X$ be a set, $\alpha, \alpha' \subset \mathcal{P}^2(X)$ and $A \subset \mathcal{P}(X)$. Then

1. $\text{sec} A$ is a stack

2. $\alpha \subset \alpha' \Rightarrow \overline{\text{stack}}_{<_{co}} \alpha \subset \overline{\text{stack}}_{<_{co}} \alpha'$

3. $\overline{\text{stack}}^2_{<_{co}} \alpha = \overline{\text{stack}}_{<_{co}} \alpha$
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Proof. The first is trivial. For the second, we suppose \( A \in \overline{\text{stack}}_{<_{co}} \alpha \) then there is a \( B \in \alpha \) for which \( A <_{co} B \). Since \( \alpha \subset \alpha' \) we have \( A \in \overline{\text{stack}}_{<_{co}} \alpha' \). The last statement is proven as follows: \( \supset \) is a direct consequence of the previous. If \( A \in \overline{\text{stack}}_{<_{co}} ^2 \) there is a \( B \in \text{stack}_{<_{co}} \) such that \( A <_{co} B \). Hence we also have \( C \in \alpha \) with \( A <_{co} B <_{co} C \), so \( A <_{co} C \) and \( A \in \overline{\text{stack}}_{<_{co}} \alpha \), hence \( \subset \) also holds.

Besides \(<\) and \(<_{co}\) relations, we will also need the concept of star-refinement.

For any collection \( \mathcal{A} \) of subsets of a set \( X \) we define the star of a subset \( B \) of \( X \) with respect to \( \mathcal{A} \) as \( \text{St}(B, \mathcal{A}) = \bigcup \{ A \in \mathcal{A} \mid A \cap B \neq \emptyset \} \), in a similar way we define the costar of \( B \) with respect to \( \mathcal{A} \) as \( \text{coSt}(B, \mathcal{A}) = \bigcap \{ A \in \mathcal{A} \mid A \cup B \neq X \} \).

Let \( \mathcal{U}, \mathcal{V} \) be covers of a set \( X \) then we say that \( \mathcal{U} \) star-refines \( \mathcal{V} \) and write \( \mathcal{U} <<^* \mathcal{V} \) if and only if \( \{ \text{St}(U, \mathcal{U}) \mid U \in \mathcal{U} \} < \mathcal{V} \). We shall also write \( \mathcal{U} <^* \mathcal{V} \) whenever \( \{ \text{St}\{x\}, \mathcal{U} \} \mid x \in X \} < \mathcal{V} \).

Finally, we will sometimes refer to the operation \( \land \) on covers of a set, in this case we mean that, for two covers \( \mathcal{U} \) and \( \mathcal{V} \), \( \mathcal{U} \land \mathcal{V} \) is the cover \( \{ U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V} \} \).

5.1.2 Classical results

We start by recalling some definitions and results from [31]. We introduce three isomorphic categories which consist of pre-nearness spaces and are described by uniform covers, near collections and small collections.

**Definition 5.1.2.** A pre-nearness space of uniform covers \((X, \mu)\) is a set \( X \), together with a set \( \mu \subset \mathcal{P}^2(X) \) of “uniform covers” satisfying:

(U1) if \( \mathcal{U} < \mathcal{V} \) and \( \mathcal{U} \in \mu \) then \( \mathcal{V} \in \mu \)

(U2) if \( \mathcal{U} \in \mu \) then \( \cup \mathcal{U} = X \)

(U3) \( \emptyset \neq \mu \neq \mathcal{P}^2(X) \)

We write \( \overline{X} \) for \((X, \mu)\) and \( \mu_{\overline{X}} \) for \( \mu \). A morphism \( f : \overline{X} \to \overline{Y} \) between two pre-nearness spaces of uniform covers is a map \( f : X \to Y \) such that:

\[ \forall \mathcal{U} \in \mu_{\overline{Y}} : f^{-1}(\mathcal{U}) = \{ f^{-1}(U) \mid U \in \mathcal{U} \} \in \mu_{\overline{X}} \]
or equivalently (using (U1))
\[ \forall U \in \mu_X^Y : \exists V \in \mu_X^X : f(V) = \{ f(V) | V \in V \} < U \]
Thus we obtain the category \( \mathbf{PNear} \) of pre-nearness spaces of uniform covers.

**Definition 5.1.3.** A pre-nearness space of near collections \((X, \xi)\) is a set \(X\), together with a set \(\xi \subset P^2(X)\) of “near collections” satisfying:

(N1) if \(A <_{co} B\) and \(B \in \xi\) then \(A \in \xi\)
(N2) if \(\cap A \neq \emptyset\) then \(A \in \xi\)
(N3) \(\emptyset \neq \xi \neq P^2(X)\)

We write \(\underline{X}\) for \((X, \xi)\) and \(\xi_X\) for \(\xi\). A morphism \(f : \underline{X} \to \underline{Y}\) between two pre-nearness spaces of near collections is a map \(f : X \to Y\) such that:
\[ \forall A \in \xi_X : f^{-1}(A) = \{ f^{-1}(A) | A \in A \} \in \xi_Y \]
Thus we obtain the category \(\mathbf{PNear}\) of pre-nearness spaces of near collections.

**Definition 5.1.4.** A pre-nearness space of small collections \((X, \gamma)\) is a set \(X\), together with a set \(\gamma \subset P^2(X)\) of “small collections” satisfying:

(S1) if \(A <_{co} B\) and \(A \in \gamma\) then \(B \in \gamma\)
(S2) \(\forall x \in X : \{\{x\}\} \in \gamma\)
(S3) \(\emptyset \neq \gamma \neq P^2(X)\)

We write \(\underline{X}\) for \((X, \gamma)\) and \(\gamma_X\) for \(\gamma\). A morphism \(f : \underline{X} \to \underline{Y}\) between two pre-nearness spaces of near collections is a map \(f : X \to Y\) such that:
\[ \forall A \in \gamma_X : f(A) = \{ f(A) | A \in A \} \in \gamma_Y \]
Thus we obtain the category \(\mathbf{PNear}\) of pre-nearness spaces of small collections.

These definitions imply the following well-known theorem [31].
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Theorem 5.1.5. The categories $^\circ\text{PNear}$, $^\bullet\text{PNear}$ and $^*\text{PNear}$ are pairwise isomorphic.

From now on we will no longer make any difference between the categories $^\circ\text{PNear}$, $^\bullet\text{PNear}$ and $^*\text{PNear}$. We will use the category $\text{PNear}$ instead and refer to the appropriate structure of an object $X$ by $\mu_X$, $\xi_X$ and $\gamma_X$. Conversions between the different pre-nearness structures are given by the following rules as indicated in [31].

\[
\begin{align*}
A \in \mu_X \iff & \forall B \in \xi_X : A \cap \text{sec} \ B \neq \emptyset \\
A \in \xi_X \iff & \forall B \in \mu_X : B \cap \text{sec} \ A \neq \emptyset \\
A \in \xi_X \iff & \text{sec} \ A \in \gamma_X \\
A \in \gamma_X \iff & \text{sec} \ A \in \xi_X \\
A \in \mu_X \iff & \forall B \in \gamma_X : A \cap \text{stack} \ B \neq \emptyset \\
A \in \gamma_X \iff & \forall B \in \mu_X : B \cap \text{stack} \ A \neq \emptyset
\end{align*}
\]

5.2 The category UPNear

Since our final aim is to describe a uniform counterpart to the the category of the zero-dimensional closure spaces, in such a way that these spaces possess a well-behaved completion, we will introduce the category of uniform pre-nearness spaces. We will follow the ideas of Tukey [45] and Bourbaki [17] and we will describe a uniform pre-nearness space by means of a set of uniform covers $\mu_X$, a set of entourages $\mathcal{D}_X$ and a family of pseudometrics $\Psi_X$.

5.2.1 Equivalent descriptions of UPNear-objects

Proposition 5.2.1. Let $\underline{X}$ be a pre-nearness space. The following are equivalent:

1. if $U \in \mu_X$ then there is a $V \in \mu_X$ such that $V \ll^* U$,
   i.e. $\{\text{St}(V, V) | V \in V\} < U$

2. if $A \not\in \xi_X$ then there is a $B \not\in \xi_X$ such that $\{\text{coSt}(B, B) | B \in B\} <_{\text{co}} A$
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3. If \( \text{sec} \, \mathcal{A} \not\in \gamma_X \) then there is a \( \text{sec} \, \mathcal{B} \not\in \gamma_X \) such that
   \[ \{ \text{coSt}(B, B) | B \in \mathcal{B} \} <_{co} \mathcal{A} \]

Proof. First we note that \( \mathcal{A} \not\in \xi_X \) \( \iff \{ X \setminus A | A \in \mathcal{A} \} \in \mu_X \) and \( \mathcal{A} \in \mu_X \) \( \iff \{ X \setminus A | A \in \mathcal{A} \} \not\in \xi_X \). This is a result from [31], where collections \( \mathcal{A} \not\in \xi_X \) are called “far”-collections and are also used to characterize pre-nearness spaces. With this one proves the following:

1\( \Rightarrow \) 2 Let \( \mathcal{A} \not\in \xi_X \), then \( \{ X \setminus A | A \in \mathcal{A} \} \in \mu_X \), hence it has a star-refinement \( \mathcal{U} \in \mu_X \). Let \( \mathcal{B} = \{ X \setminus U | U \in \mathcal{U} \} \not\in \xi_X \). For this \( \mathcal{B} \) we have that
   \[ \{ \text{coSt}(B, B) | B \in \mathcal{B} \} <_{co} \mathcal{A} \].

2\( \Rightarrow \) 1 Let \( \mathcal{U} \in \mu_X \), then \( \{ X \setminus U | U \in \mathcal{U} \} \) is not in \( \xi_X \), hence there exists a costar-co-refinement \( \mathcal{A} \not\in \xi_X \). Let \( \mathcal{V} = \{ X \setminus A | A \in \mathcal{A} \} \in \mu_X \). With this choice of \( \mathcal{V} \), it follows that \( \mathcal{V} <_{<} \mathcal{U} \).

2\( \Leftrightarrow \) 3 Obvious since \( \mathcal{A} \in \xi_X \) \( \iff \text{sec} \, \mathcal{A} \in \gamma_X \).

\( \square \)

Definition 5.2.2. A pre-nearness space \( X \) satisfying one, and hence all conditions of Proposition 5.2.1, will be called a \textbf{uniform pre-nearness space}, i.e. it is a pre-nearness space for which the star-refinement axiom

\[ (U^*) \forall \mathcal{U} \in \mu_X : \exists \mathcal{V} \in \mu_X : \mathcal{V} <_{<}^{*} \mathcal{U} \]

holds. The \textbf{category of uniform pre-nearness spaces} is the full subcategory of \( \text{PNear} \) given by the uniform pre-nearness spaces and will be written as \( \text{UPNear} \).

With the usual forgetful functor \( U : \text{UPNear} \to \text{Set} \), \( \text{UPNear} \) is a well-fibred topological construct. A \( U \)-structured source \( (f_i : X \to U X_i)_{i \in I} \) is lifted by means of the initial uniform pre-nearness structure on \( X \) given by \( \text{stack}_{<}(f_i^{-1}(U_i)) \cup U_i \in \mu_X, i \in I \).

Inspired by the classical theories of Tukey [45] and Bourbaki [17] we introduce the following two categories.

Definition 5.2.3. A \( \Delta \)-uniform pre-nearness space \( (X, \mathcal{D}) \) is a set \( X \), together with a nonempty set \( \mathcal{D} \subset \mathcal{P}(X \times X) \) of “entourages” satisfying

\[ f \]
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\((\Delta 1)\) \(\forall D \in \mathcal{D} : \Delta_X \subset D\)

\((\Delta 2)\) \(\forall D \in \mathcal{D} : D \subset E \Rightarrow E \in \mathcal{D}\)

\((\Delta 3)\) \(\forall D \in \mathcal{D} : \exists E \in \mathcal{D} : E \circ E \subset D\)

\((\Delta 4)\) \(\forall D \in \mathcal{D} : \exists E \in \mathcal{D} : E = E^{-1} \subset D\)

where we use the usual notations \(\Delta_X = \{(x,x) \in X \times X | x \in X\}\), \(E \circ D = \{(x,y) \in X \times X | \exists z \in X : (x,z) \in D\) and \((z,y) \in E\}\), \(D^{-1} = \{(x,y) \in X \times X | (y,x) \in D\}\).

We write \(\mathcal{X}\) for \((X, \mathcal{D})\) and \(\mathcal{D}_X\) for \(\mathcal{D}\). A morphism \(f : X \to Y\) between two \(\Delta\)-uniform pre-nearness spaces is a map \(f : X \to Y\) such that:

\(\forall D \in \mathcal{D}_Y : (f \times f)^{-1}(D) \in \mathcal{D}_X\)

or equivalently

\(\forall D \in \mathcal{D}_Y : \exists E \in \mathcal{D}_X : f \times f(E) \subset D\)

Thus we obtain the category \(\Delta_{UPNear}\) of \(\Delta\)-uniform pre-nearness spaces.

For a uniform pre-nearness space \(\mathcal{X}\), a subcollection \(E\) of \(\mathcal{D}_X\) will be called a base for \(\mathcal{D}_X\) if and only if \(\mathcal{D}_X = \text{stack } E\). It is obvious that any collection \(E \subset \mathcal{P}(X \times X)\) which satisfies \((\Delta 2)\), \((\Delta 3)\) and \((\Delta 4)\) is a base for a \(\Delta\)-uniform pre-nearness structure on \(X\).

With the usual forgetful functor \(U : \Delta_{UPNear} \to \text{Set}\), \(\Delta_{UPNear}\) is a well-fibred topological construct. A \(U\)-structured source \((f_i : X \to UX_i)_{i \in I}\) is lifted by means of the initial \(\Delta\)-uniform pre-nearness structure on \(X\) given by stack \(\{(f_i \times f_i)^{-1}(D_i) | D_i \in \mathcal{D}_X, i \in I\}\). From the existence of initial lifts we obtain suprema \((\vee_{i \in I} D_i)_{i \in I}\), in occurrence \(\vee_{i \in I} D_i = \cup_{i \in I} D_i\), which we will need later on.

**Remark 5.2.4.** In the classical theory of uniform spaces a uniformity on \(X\) is a filter \(D\) on \(X \times X\) satisfying \((\Delta 1)-(\Delta 3)\) and for which \(D \in \mathcal{D}\) implies \(D^{-1} \in \mathcal{D}\). Note that we did not only drop the axiom about finite intersections (a \(\Delta_{UPNear}\) structure is a stack, not a filter) but that we also replaced the existence of inverse elements by the stronger condition \((\Delta 4)\) about symmetric elements. This is needed because we want to have a symmetric base for every \(\Delta\)-uniform pre-nearness structure.

**Definition 5.2.5.** A pseudometric on a set \(X\) is a map \(\psi : X \times X \to \mathbb{R}^+\) such that:
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1. \( \forall x \in X : \psi(x, x) = 0 \)
2. \( \forall x, y \in X : \psi(x, y) = \psi(y, x) \)
3. \( \forall x, y, z \in X : \psi(x, y) \leq \psi(x, z) + \psi(z, y) \)

Definition 5.2.6. A uniform pre-nearness space of pseudometrics \((X, \Psi)\) is a set \(X\) together with a set \(\Psi\) of pseudometrics satisfying:

\[ (\forall \epsilon > 0 : \exists \psi_1 \in \Psi : \exists \delta > 0 : \psi_1(x, y) < \delta \Rightarrow \psi_2(x, y) < \epsilon) \Rightarrow \psi_2 \in \Psi \]

We write \(\overline{X}\) for \((X, \Psi)\) and \(\Psi_X\) for \(\Psi\). A morphism \(f : \overline{X} \to \overline{Y}\) between two uniform pre-nearness spaces of pseudometrics is a map \(f : X \to Y\) such that:

\[ \forall \psi \in \Psi_Y : \psi \circ (f \times f) \in \Psi_X \]

Thus we obtain the category \(\text{PsUPNear}\) of uniform pre-nearness spaces of pseudometrics.

With the usual forgetful functor \(U : \text{PsUPNear} \to \text{Set}\), \(\text{PsUPNear}\) is a well-fibred topological construct. A \(U\)-structured source \((f_i : X_i \to \overline{X}_i)_{i \in I}\) is lifted by means of the initial uniform pre-nearness of pseudometrics on \(X\) given by all pseudometrics \(\psi\) on \(X\) satisfying the following condition

\[ \forall \epsilon > 0 : \exists i \in I, \psi_1 \in \Psi_{X_i} : \exists \delta > 0 : \psi_1(f_i(x), f_i(y)) < \delta \Rightarrow \psi(x, y) < \epsilon \]

Next we will prove that the categories \(\text{UPNear}, \Delta\text{UPNear}\) and \(\text{PsUPNear}\) are pairwise isomorphic. We first consider \(\text{UPNear}\) and \(\Delta\text{UPNear}\).

Proposition 5.2.7. If \((X, \mu)\) is an uniform pre-nearness space then

\[ D\mu = \{ D \subset X \times X | \exists U \in \mu : D_U \subset D \} \]

with \(D_U = \cup\{ u \times u | u \in U \} \), is a \(\Delta\)-uniform pre-nearness structure on \(X\). If \((X, D)\) is a \(\Delta\)-uniform pre-nearness space then

\[ \mu D = \{ U \text{ cover} \exists D \in D : U_D < U \} \]

with \(U_D = \{ D[x] | x \in X \}\) and \(D[x] = \{ y \in X | (x, y) \in D \} \), is a uniform pre-nearness structure. Moreover the Galois correspondence

\[ \mu D \mu = \mu \text{ and } D \mu D = D \]

holds.
Proof. Let \((X, \mu)\) be an uniform pre-nearness cover space. We only show \((\Delta 3)\), the other axioms are easy verifications. If \(D \in \mathcal{D}\) there is a \(U \in \mu\) for which \(D_U \subset D\) and an \(V \in \mu\) which star-refines \(U\): \(V \ll \ast U\). Then \(D_V \in \mathcal{D}_\mu\) and \(D_V \circ D_V \subset D_U \subset D\).

We shall now turn to the second statement, we prove only \((U^*)\). Let \(E\) be a symmetric base of \(D\). Given a \(U \in \mu\), we choose a \(D \in D\) so that \(U < D\). Hence \(U_E < U\). Repeating this with \(U_E\) instead of \(U\) we find an \(E' \in E\) so that \(U < U_E < U\) which implies \(U_E < U\), thus we have proven \((U^*)\).

Finally, for the last two statements. Since \(U < U_D\), \(\mu D \subset \mu\). On the other hand, starting with \(U \in \mu\) we have a \(V \in \mu\) such that \(V \ll \ast U\), implying \(V < U\). Hence \(\{St(\{x\}, U)|x \in X\} < U\) and \(\{D_U[x]|x \in X\} < U\). So \(U_D < U\), therefore \(U \in \mu D\) and \(\mu D = \mu\).

For a symmetric \(E\) such that \(E < E\), \(\mu D \subset \mu\). Hence \(D \subset D_{\mu D}\). Conversely, if \(U_D\) is in \(\mathcal{D}\) then \(U_D \in \mu\). Thus we get an \(U_E\) which star-refines \(U_D\), hence also \(U_E < U_D\). Therefore \(E \subset D_{U_D}\) and \(D_{U_D} \subset D\) and therefore \(D_{\mu D} = D\).

**Theorem 5.2.9.** \(\Delta\text{UPNear}\) and \(\text{UPNear}\) are isomorphic categories.

**Proof.** Define the following functors:

\[
\begin{align*}
F_\mu : \Delta\text{UPNear} & \rightarrow \text{UPNear} \\
(X, \mathcal{D}) & \mapsto (X, \mu \mathcal{D}) \\
f & \mapsto f
\end{align*}
\]

\[
\begin{align*}
F_\mathcal{D} : \text{UPNear} & \rightarrow \Delta\text{UPNear} \\
(X, \mu) & \mapsto (X, \mu \mathcal{D}) \\
f & \mapsto f
\end{align*}
\]
the result follows by applying the previous propositions.

Next we will show that $\Delta\text{UPNear}$ and $\text{PsUPNear}$ are also isomorphic categories. We first recall a well-known result from the theory of uniform spaces.

**Theorem 5.2.10.** If $\psi$ is a pseudometric on a set $X$, then $U_\psi = \{ \psi^{-1}([0, \epsilon]) | \epsilon > 0 \}$ is a uniformity on $X$. Conversely, a uniform space $(X, \mathcal{U})$ with a countable base, is pseudometrizable, i.e. there is a pseudometric $\psi$ on $X$ such that $\mathcal{U} = U_\psi$.

**Theorem 5.2.11.** In $\Delta\text{UPNear}$ each $\Delta$-uniform pre-nearness structure is the $\Delta\text{UPNear}$-supremum of a family of uniformities, induced by a family of pseudometrics.

**Proof.** Let $(X, \mathcal{D})$ be a $\Delta$-uniform pre-nearness space and $D \in \mathcal{D}$, symmetric. We can choose a decreasing sequence $(D_n)_n$ of symmetrical elements of $\mathcal{D}$ such that:

$$D_1 = D, D_2 \subset D_1, D_3 \subset D_2, \ldots, D_{n+1} \subset D_n, \ldots$$

$\{D_n\}$ is a countable base for a uniformity $U_D$ on $X$ which can be described by a pseudometric $\psi_D$. Because $1_X : (X, \mathcal{D}) \to (X, \mathcal{U})$ is a $\Delta\text{UPNear}$-morphism we can say that $\bigvee_{D \in \mathcal{D}} U_D \subset D$ holds for the $\Delta\text{UPNear}$-supremum. Conversely for each $D \in \mathcal{D}$ we have a symmetric $E \in \mathcal{D}$ for which $E \subset D$, so $D \in \mathcal{U}_E$ thus $\mathcal{D} \subset \bigcup_{D \in \mathcal{D}} U_D = \bigvee_{D \in \mathcal{D}} U_D$. Therefore $\mathcal{D} = \bigvee_{D \in \mathcal{D}} U_D$.

From now on we will write $\mathcal{D}\Psi$ for the $\Delta$-uniform pre-nearness structure $\bigcup_{\psi \in \Psi} U_\psi$ generated by a family of pseudometrics $\Psi$.

We will now associate with a $\Delta$-uniform pre-nearness space a unique collection of pseudometrics. In the classical case of a uniform space $X$ this is done by considering the collection $\Psi$ of all uniform continuous pseudometrics $\psi : X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the usual uniform space on $\mathbb{R}^+$. In the case of uniform pre-nearness spaces the situation differs from the classical one as is shown in the next example.

**Example 5.2.12.** [The $\Delta\text{UPNear}$-space $D_2$ generated by a pseudometric $\psi$, such that $\psi : D_1 \times D_1 \to \mathbb{R}^+$ is not a $\Delta\text{UPNear}$-morphism.]

Let $D_2$ be the discrete $\Delta\text{UPNear}$-space on two points $\{0, 1\}$. It has a base $\{\Delta_{\{0,1\}}\}$ and is induced by the pseudometric $\psi(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$. A base for the $\Delta$-uniform pre-nearness structure of the space $D_2 \times D_2$ is given by
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\{(pr_1 \times pr_1)^{-1}(\Delta_{(0,1)}), (pr_2 \times pr_2)^{-1}(\Delta_{(0,1)})\}. \text{ Hence for the point } \bar{a} = ((0,0),(0,1)) we obtain: \( pr_1 \times pr_1(\bar{a}) = (0,0) \in \Delta_{(0,1)} \) but \( \psi \times \psi(\bar{a}) = (0,1) \notin \Delta_{(0,1)} \), so \( (pr_1 \times pr_1)^{-1}(\Delta_{(0,1)}) \not\subset (\psi \times \psi)^{-1}(V) \), where \( V = \{(x,y) \in \mathbb{R}^+ | |x-y| < \epsilon \} \) and \( 0 < \epsilon < 1 \). Replacing \( \bar{a} \) with \( \bar{b} = ((0,0),(1,0)) \) and \( pr_1 \) with \( pr_2 \) the same argument yields \( (pr_2 \times pr_2)^{-1}(\Delta_{(0,1)}) \not\subset (\psi \times \psi)^{-1}(V) \). Hence \( \psi : D_x \times D_y \to \mathbb{R}^+ \) is not a \( \Delta \text{-UPNear-morphism} \). ▶

In order to construct a unique family of pseudometrics associated with a given \( \Delta \)-uniform pre-nearness space we prove the next proposition.

**Proposition 5.2.13.** Consider a \( \Delta \)-uniform pre-nearness space \( (X,D) \). Let

\[ \Psi D = \{ \psi \text{ pseudometric} | \forall \epsilon > 0 : \psi^{-1}([0,\epsilon]) \in D \} \]

Then \( \Psi D \) is the largest collection of pseudometrics that induces \( D \).

**Proof.** Choose a collection \( \Psi' \) of pseudometrics such that \( D = \cup_{\psi \in \Psi'} U_{\psi} \), where \( U_{\psi} \) is the uniformity induced by \( \psi \). We have for an \( \epsilon > 0 \) and \( \psi \in \Psi' : \psi^{-1}([0,\epsilon]) \in U_{\psi} \subset D \), so \( \Psi' \subset \Psi D \). Therefore \( D \subset \cup_{\psi \in \Psi D} U_{\psi} \) and of course if \( \psi \in \Psi D \Rightarrow U_{\psi} \subset D \), so \( D = \cup_{\psi \in \Psi D} U_{\psi} \). □

**Remark 5.2.14.** Let \( X = (X,D) \). Although for a pseudometric \( \psi \) in \( \Psi D \), \( \psi : X \times X \to \mathbb{R}^+ \) is not always a \( \Delta \text{-UPNear-morphisms} \), we can say that for any \( a \in X \) and \( \psi \in \Psi D \), \( \psi_a : X \to \mathbb{R}^+ : x \mapsto \psi(x,a) \) is a \( \Delta \text{-UPNear-morphisms} \). To see this we first consider \( \psi_a : (X,U_{\psi}) \to \mathbb{R}^+ \) which is a \( \Delta \text{-UPNear-morphisms} \). Since \( U_{\psi} \subset D \), \( \psi_a : X \to \mathbb{R}^+ \) is a \( \Delta \text{-UPNear-morphisms} \).

**Proposition 5.2.15.** If \( (X,\Psi) \) is an uniform pre-nearness space of pseudometrics then

\[ D_{\Psi} = \vee_{\psi \in \Psi} U_{\psi} \]

where \( U_{\psi} \) is the uniform structure induced by the pseudometric \( \psi \), is a \( \Delta \)-uniform pre-nearness structure on \( X \). If \( (X,D) \) is a \( \Delta \)-uniform pre-nearness space then

\[ \Psi D = \{ \psi \text{ pseudometric} | \forall \epsilon > 0 : \psi^{-1}([0,\epsilon]) \in D \} \]

yields a uniform pre-nearness of pseudometrics. Moreover the Galois correspondence

\[ \Psi D \Psi = \Psi \text{ and } D_{\Psi D} = D \]

holds.
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Proof. The first part follows immediately from Theorem 5.2.11.

For the second part, let \( \psi_2 \) be a pseudometric on \( X \) satisfying
\[
\forall \epsilon > 0 : \exists \psi_1 \in \Psi D : \exists \delta > 0 : \psi_1(x, y) < \delta \Rightarrow \psi_2(x, y) < \epsilon
\]
Choose \( \epsilon > 0 \), then there is a \( \psi_1 \in \Psi D \) and a \( \delta > 0 \) such that \( \psi_1^{-1}([0, \delta]) \subset \psi_2^{-1}([0, \epsilon]) \), because \( \psi_1^{-1}([0, \delta]) \in D \) we have that \( \psi_2^{-1}([0, \epsilon]) \in D \). Therefore \( \psi_2 \in \Psi D \). Hence \( (X, \Psi D) \) is a uniform pre-nearness of pseudometrics.

To prove the Galois correspondence we first observe that, since \( \Psi D \) is the largest collection of pseudometrics which induces \( D \) it follows that \( D \Psi = D \). If \( \psi \in \Psi D \Psi \) then
\[
\forall \epsilon > 0 : \psi^{-1}([0, \epsilon]) \in D \Psi = \cup_{\psi \in \Psi} U \psi
\]
\[
\Rightarrow \forall \epsilon > 0 : \exists \psi' \in \Psi : \psi'^{-1}([0, \epsilon]) \in U \psi',
\]
\[
\Rightarrow \forall \epsilon > 0 : \exists \psi' \in \Psi : \exists \delta > 0 : \psi'^{-1}([0, \delta]) \subset \psi^{-1}([0, \epsilon])
\]
\[
\Rightarrow \forall \epsilon > 0 : \exists \psi' \in \Psi D : \exists \delta > 0 : \psi'(x, y) < \delta \Rightarrow \psi(x, y) < \epsilon
\]
Conversely we have that \( \psi \in \Psi \) implies, for every \( \epsilon > 0 \), \( \psi^{-1}([0, \epsilon]) \in U \psi \subset D \Psi \) so \( \psi \in \Psi D \Psi \).

Proposition 5.2.16. \( f : (X, \Psi) \to (Y, \Phi) \) is a \( \text{PsUPNear} \)-morphism if and only if \( f : (X, D \Psi) \to (Y, D \Phi) \) is a \( \text{UPNear} \)-morphism.

Proof. If \( \varphi^{-1}([0, \epsilon]) \in D \Phi \) then \( (f \times f)^{-1}(\varphi^{-1}([0, \epsilon])) = (\varphi \circ f \times f)^{-1}([0, \epsilon]) \in D \Psi \). Conversely, it is obvious that \( \varphi \circ f \times f \) is a pseudometric if \( \varphi \in \Phi \). We prove that \( \varphi \circ f \times f \in \Psi \). For \( \epsilon > 0 \) we choose a \( \delta > 0 \) and a \( \psi \in \Psi \) such that \( \psi^{-1}([0, \delta]) \subset (f \times f)^{-1}(\varphi^{-1}([0, \epsilon])) \in D \Psi \). If \( \psi(x, y) < \delta \) this implies \( \varphi(f(x), f(y)) < \epsilon \), therefore \( \varphi \circ f \times f \in \Psi \).

Theorem 5.2.17. \( \text{UPNear} \) and \( \text{PsUPNear} \) are isomorphic categories.

Proof. Define the following functors:
\[
G_\Psi : \text{UPNear} \to \text{PsUPNear}
\]
\[
G_D : \text{PsUPNear} \to \text{UPNear}
\]
\[
(X, D) \mapsto (X, \Psi D)
\]
\[
f \mapsto f
\]
\[
(X, \Psi) \mapsto (X, D \Psi)
\]
\[
f \mapsto f
\]
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The result follows by applying the previous propositions.

We shall no longer make any difference between \textsc{UPNear}, \textsc{PsUPNear} or \textsc{ΔUPNear}, instead we’ll use \textsc{UPNear} and write $X$ for a \textsc{UPNear}-object and refer to one of its defining structures as $\mu_X, \gamma_X, \xi_X, D_X$ or $\Psi_X$. In the remaining part of this section we will describe some useful conversions between these different structures.

**Proposition 5.2.18.** Let $X$ be a uniform pre-nearness space. A stack $G$ is a small stack ($G \in \gamma_X$) if and only if satisfies the condition

$$\forall D \in D_X : \exists F \in G : F \times F \subset D$$

**Proof.** Let $G$ be a stack. If for any $D \in D_X$ there is an $F \in G$ for which $F \times F \subset D$ then there is an $x \in X$ such that $F \subset D[x]$. Hence for every $U_D \in \mu_X = \mu_D$ we have an $F \in G$ and a $U = D[x] \in U_D$ for which $F \subset U$. Conversely if for every $U \in \mu_X = \mu_D$ there is a $F \in G$ and a $U \in U$ such that $F \subset U$, then $F \times F \subset D_U$.

**Proposition 5.2.19.** Let $X$ be a uniform pre-nearness space. Define

$$\mu_D_X = \{U_D | D \in D_X\}$$

$$\xi_D_X = \{A \subset P(X) | \forall U_D \in \mu_D_X : U_D \cap \text{sec} A \neq \emptyset\}$$

then: stack $< \mu_D_X = \mu_X$ and stack $< \co \xi_D_X = \xi_X$.

**Lemma 5.2.20.**

$$A \in \xi_D_X \iff A \subset P(X), \forall U_D \in \mu_D_X : U_D \cap \text{sec} A \neq \emptyset$$

$$\iff A \subset P(X), \forall D \in D : \exists x \in X : D[x] \in \text{sec} A$$

$$\iff A \subset P(X), \forall D \in D : \exists x \in X : \forall A' \in A : D[x] \cap A' \neq \emptyset$$

**Proof of the proposition.** The first statement is true by definition of $\mu_D_X = \mu_X$. For the latter, $\iff$ is obvious by the construction of $\xi_X$.

$$A \in \text{stack}_< \xi_D_X$$

$$\iff \exists A' \in \xi_D_X : A <_\text{co} A'$$

$$\iff \exists A' \subset P(X) : \forall D \in D : \exists x \in X : \forall A' \in A' : D[x] \cap A' \neq \emptyset, A <_\text{co} A'$$

For $U > U_D$ in $\mu_X$, we choose an $x \in X$ so that $\forall A' \in A' : D[x] \cap A' \neq \emptyset, A <_\text{co} A'$, we also have that there is a $U \in U$ such that $D[x] \subset U$, therefore $U \cap \text{sec} A' \neq \emptyset$ for any $A' \in A$ and thus $U \cap \text{sec} A' \neq \emptyset$, so $A' \in \xi_X$ implying $A \in \xi_X$. 

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Proposition 5.2.21. With the same notations and
\[ \Xi = \{ \text{sec} (\{D[x_D]|D \in D_X\}) | x \in X^{D_X} \} \]
we have: \( \underline{\text{stack}}_{<co} \Xi = \xi_X \) (by \( x_D \) we mean the \( D \)th projection of \( x \)).

Proof. For \( x \in X^{D_X} \) and \( D \in D_X \) we have that there is an \( x_D \) so that for all \( A \in \text{sec} \{D[x_D]|D \in D_X\} \) obviously \( A \cap D[x_D] \neq \emptyset \), therefore by the above lemma: \( \Xi \subset \xi_D \). Choose \( A \in \xi_{D_X} \), for any \( D \in D \) we have an \( x_D \in X \) such that for every \( A \in A \), \( D[x] \cap A \neq \emptyset \). Define \( x = (x_D)_{D \in D_X} \in X^{D_X} \), then: \( A \in A \) implies \( D[x_D] \cap A \neq \emptyset \) for every \( D \in D_X \). Hence \( A \in \text{sec} (\{D[x_D]|D \in D_X\}) \). Thus \( \mathcal{A} \subset \text{sec} (\{D[x_D]|D \in D_X\}) \) and \( A \subset \text{sec} (\{D[x_D]|D \in D_X\}) \). From this we get \( A \in \underline{\text{stack}}_{<co} \Xi \) and \( \xi_{D_X} \subset \underline{\text{stack}}_{<co} \Xi \), hence
\[ \Xi \subset \xi_{D_X} \subset \underline{\text{stack}}_{<co} \Xi \]
\[ \Rightarrow \underline{\text{stack}}_{<co} \Xi \subset \underline{\text{stack}}_{<co} \xi_{D_X} \subset \underline{\text{stack}}_{<co} \Xi \]
applying the previous proposition and simplifying using the Lemma 5.1.1 we get \( \underline{\text{stack}}_{<co} \Xi = \xi_X \).

Proposition 5.2.22. Let \( X \) be a uniform pre-nearness space.
\[ \underline{\text{stack}}_{<co} \{ \text{sec} \ A_i | i \in I \} = \xi_X \leftrightarrow \underline{\text{stack}}_{<co} \{ \text{sec} \ A_i | i \in I \} = \gamma_X \]

Proof.
\[ \Rightarrow \ A \in \gamma_X \Leftrightarrow \text{sec} \ A \in \xi_X \]
\[ \Rightarrow \ \exists i \in I : \text{sec} \ A \subset \text{sec} \ A_i \]
\[ \Rightarrow \ \exists i \in I : A_i \subset A \]
\[ \Rightarrow \ A \in \underline{\text{stack}}_{<co} \{ A_i | i \in I \} \]
\[ \Leftarrow \ A \in \xi_X \Leftrightarrow \text{sec} \ A \in \gamma_X \]
\[ \Rightarrow \ \exists i \in I : \text{sec} \ A_i \subset \text{sec} \ A \]
\[ \Rightarrow \ \exists i \in I : \text{sec}^2 \ A \subset \text{sec} \ A_i \]
\[ \Rightarrow \ \exists i \in I : A \subset \text{sec} \ A_i \]
\[ \Rightarrow \ A \in \underline{\text{stack}}_{<co} \{ \text{sec} \ A_i | i \in I \} \]

Proposition 5.2.23. Let \( X \) be a uniform pre-nearness space.
\[ \xi_X = \underline{\text{stack}}_{<co} \{ \text{sec} (\{D[x_D]|D \in D_X\}) | x \in X^{D_X} \} \]
\[ \gamma_X = \underline{\text{stack}}_{<co} \{ \{D[x_D]|D \in D_X\} | x \in X^{D_X} \} \]
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Proof. Follows from the above results.

5.2.2 Unif as a subcategory of UPNear

From the definition of uniform pre-nearness spaces and the morphisms between them, it is obvious that the category Unif of uniform spaces and uniform continuous maps is a full subcategory of UPNear. In fact, the following proposition is an immediate corollary of Theorem 5.2.11.

**Proposition 5.2.24.** Unif is an initially dense subcategory of UPNear.

As a full and initially dense subcategory of a well-fibred topological construct, Unif is in fact bicoreflective in UPNear.

**Proposition 5.2.25.** Unif is a bicoreflective subcategory of UPNear.

Proof. We construct the bicoreflection of a UPNear space $X$ as follows. By Theorem 5.2.11 $X$ is the UPNear-supremum of a family of uniform spaces $(X_i)_{i \in I}$ on $X$. Denote by $X^u$ the Unif-supremum of $(X_i)_{i \in I}$. Since for any $i \in I$ the map $1_X : X^u \to X_i$ is a UPNear-morphism and $(1_X : X \to X_i)_{i \in I}$ is initial in UPNear, $1_X : X^u \to X$ is a UPNear-morphism. Let $Y$ be a uniform space and $f : Y \to X$ a UPNear-morphism, we obtain the following diagram:

$$
\begin{array}{c}
Y \\
\downarrow^f \\
X^u \\
\downarrow_{1_X} \\
X_i \\
\uparrow^{1_X}
\end{array}
$$

for every $i \in I$. Using the initiality of $(1_X : X^u \to X_i)_{i \in I}$ in Unif, we find that the map $f : Y \to X^u$ is the unique uniformly continuous function making the diagram commutative. Hence the bicoreflection of $X$ in Unif is given by $1_X : X^u \to X$.

Thus we have proven that in order to find the Unif-bicoreflection of a UPNear space $X$ one has to write $X$ as a UPNear-supremum of uniform spaces $(X_i)_{i \in I}$ and consider the Unif-supremum $X^u$ of the same family. Hence it follows that $D_{X^u}$ is the saturation of $D_X$ for finite intersections, that $\mu_{X^u}$ is the saturation of $\mu_X$ under finite $\land$ and that $\Psi_{X^u}$ is obtained by saturation for finite $\lor$ from $\Psi_X$.
With a uniform pre-nearness space we will now associate a closure space.

Lemma 5.2.26. Any uniform pre-nearness space $X$ satisfies the following nearness axiom:

$$\mathcal{U} \in \mu_X \Rightarrow \{\text{int}_{\mu_X}(\mathcal{U}) | \mathcal{U} \in \mathcal{U}\} \in \mu_X$$

where $\text{int}_{\mu_X}(\mathcal{U}) = \{x \in X | \{U, X \setminus \{x\}\} \in \mu_X\}$.

Proof. Take $\mathcal{U} \in \mu$ then by $(\mathcal{U}^*)$ there is a $\mathcal{V} \in \mu$ such that $\mathcal{V} \ll^* \mathcal{U}$, hence for any $\mathcal{V} \in \mathcal{V}$ there is a $\mathcal{U} \in \mathcal{U}$ for which $\text{St}(\mathcal{V}, \mathcal{V}) \subset \mathcal{U}$. Thus we have for $x \in \mathcal{V}$:

$$\mathcal{V} \ll \{\text{int}_{\mu_X}(\mathcal{U}) | \mathcal{U} \in \mathcal{U}\}.$$  

Proposition 5.2.27. Let $X$ be a uniform pre-nearness space. The following closure structures induce the same closure space.

1. The neighborhood collections $(\mathcal{V}(x))_{x \in X}$, where $\mathcal{V}(x) = \text{stack} \{\mathcal{D}[x] | \mathcal{D} \in \mathcal{D}_X\}$
2. The neighborhood collections $(\mathcal{V}(x))_{x \in X}$, where $\mathcal{V}(x) = \text{stack} \{\text{St}(\{x\}, \mathcal{U}) | \mathcal{U} \in \mu_X\}$
3. The neighborhood collections $(\mathcal{V}(x))_{x \in X}$, where $\mathcal{V}(x) = \text{stack} \{\mathcal{B}_\psi(x, \epsilon) | \psi \in \Psi_X, \epsilon > 0\}$
4. The closure operator $\text{cl}_{\mu_X}(A) = \{x \in X | \{X \setminus A, X \setminus \{x\}\} \notin \mu_X\}$
5. The closure operator $\text{cl}_{\xi_X}(A) = \{x \in X | \{A, \{x\}\} \in \xi_X\}$
6. The closure operator $\text{cl}_{\gamma_X}(A) = \{x \in X | \text{sec} \{A, \{x\}\} \in \gamma_X\}$

We will write the unique closure space defined in this way as $CX$.

Proof. Since $\mathcal{D}[x] \subset X$ it is clear that $X \in \mathcal{V}(x)$. Because $\Delta \subset \mathcal{D}, \forall \mathcal{D} \in \mathcal{D}_X$ we have that $x \in \mathcal{V}, \forall \mathcal{V} \in \mathcal{V}(x)$. Now we choose $\mathcal{V} \in \mathcal{V}(x), \mathcal{V} \subset \mathcal{W}$ then because of the stack we have $\mathcal{W} \in \mathcal{V}(x)$. Finally let $\mathcal{V} \in \mathcal{V}(x)$ then there is a $\mathcal{D} \in \mathcal{D}$ so that $\mathcal{D}[x] \subset \mathcal{V}$. There also exists an $\mathcal{E} \in \mathcal{D}$ for which $\mathcal{E} \circ \mathcal{E} \subset \mathcal{D}[x] \subset \mathcal{V}$ which implies $\forall y \in \mathcal{E}[x] : \mathcal{E}[y] \subset \mathcal{D}[x] \subset \mathcal{V}$ which implies $\forall y \in \mathcal{E}[x] : \mathcal{V}(y)$. Therefore the first family of neighborhood collections describes a closure space. We will write it’s associated closure operator as $cl_{\mathcal{D}_X}$.
The first three closure structures are equivalent by the means of the isomorphisms between $\Delta_{\text{UPNear}}$, $\text{PsUPNear}$ and $\text{UPNear}$.

It is a well-known fact from [31] that the three operators are equivalent. Moreover by the preceding lemma, they are closure operators.

Finally, we prove the equality $\text{cl}_{D_X} = \text{cl}_{\gamma_X}$:

$x \in cl_{D_X}(A) \Rightarrow \forall V \in V_x = \text{stack}\{D[x]|D \in D_X\} : A \cap V \neq \emptyset$

$\Rightarrow \forall D \in D_X : D[x] \cap A \neq \emptyset$ and, of course, $D[x] \cap \{x\} \neq \emptyset$

$\Rightarrow \forall D \in D_X : D[x] \in \text{sec} \{A, \{x\}\}$

$\Rightarrow \{D[x]|D \in D_X\} <_{co} \text{sec} \{A, \{x\}\}$

$\Rightarrow \exists y \in X^{D_X} : \{D[y_D]|D \in D_X\} <_{co} \text{sec} \{A, \{x\}\}$, choose $y_D = x$

$\Rightarrow \text{sec} \{A, \{x\}\} \in \gamma_X$

$\Rightarrow x \in cl_{\gamma_X}(A)$

For any $D \in D_X$ we choose a symmetric $E \in D_X$ with $E \circ E \subset D$. Applying the last statement to $E$ we get: $E[y_E] \cap A \neq \emptyset$ and $x \in E[y_E]$. Hence $(x, y_E) \in E$. If $z \in E[y_E]$ then $(x, y_E, z) \in E \circ E$ and $z \in D[x]$. Thus $E[y_E] \subset D[x]$ and therefore $D[x] \cap A \neq \emptyset$ for any $D \in D_X$. Thus we get $x \in cl_{D_X}(A)$.

\[\square\]

**Proposition 5.2.28.** A $\text{UPNear}$-morphism $f : X \to Y$ is also a $\text{Cl}$-morphism between the induced closure spaces.

**Proof.** A basic neighborhood of $x \in X$ is of the form $E[f(x)]$ with $E \in D_Y$ so there is a $D \in D : f \times f(D) \subset D$, so $f(D[x]) \subset E[f(x)]$. Thus $f : C_X \to C_Y$ is a $\text{Cl}$-morphism. \[\square\]

We are now able to give a functorial relation between $\text{UPNear}$ and $\text{Cl}$. 

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Proposition 5.2.29.

\[ C : \text{UPNear} \to \text{Cl} \]
\[ X \mapsto CX \]
\[ f \mapsto f \]

is a faithful functor between \text{UPNear} and \text{Cl} which preserves initial sources.

Proof. Suppose we have a structured source \((f_i : X \to X_i)_{i \in I}\) in \text{UPNear}, we write \(g_i = f_i \times f_i\). From the fact that for any \(D = g_i^{-1}(D_i), D_i \in \mathcal{D}_X:\)

\[ D[x] = \{ y \in X | (x, y) \in g_i^{-1}(D_i) \} \]
\[ = \{ y \in X | (f_i(x), f_i(y)) \in D_i \} \]
\[ = f_i^{-1}(D_i[f_i(x)]) \]

we can say that the induced closure space of an initial structure in \text{UPNear} is in fact the initial structure in \text{Cl} of the induced closure spaces. \(\square\)

Corollary 5.2.30. Let \(\underline{X}\) be a \text{UPNear}-object with \text{Unif}-bicoreflection \(X^u\). \(X^u\) induces a topological space \(TX^u\) and \(\underline{X}\) a closure space \(CX\). Then \(TX^u\) is the \text{Top}-bicoreflection of \(CX\).

We schematize the relations between the categories introduced in this chapter as follows.

\[
\begin{array}{ccc}
\text{UPNear} & \xrightarrow{\Delta} & \text{PsUPNear} \\
\text{Unif} & \xrightarrow{\text{bic}} & \text{Cl} \\
\text{Top} & \xrightarrow{\text{bic}} & \text{Cl}
\end{array}
\]

As a topological category \text{UPNear} has \(T_0\)-objects. We can characterize them as follows.

Proposition 5.2.31. Let \(\underline{X}\) be a uniform pre-nearness space. The following are equivalent:
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1. $X$ is $T_0$, in the sense of Definition 1.2.4
2. $\cap D_X = \Delta_X$
3. the underlying closure space $CX$ is Hausdorff
4. the underlying closure space $CX$ is $T_0$

Proof. Suppose $X$ is a $T_0$-object and $\cap D_X \neq \Delta_X$. There exist $x \neq y$ in $X$ such that $(x, y) \in \cap D_X$. Since the map $f : \mathbb{R}_2 \to X$ with $f(0) = x, f(1) = y$ is uniformly continuous we get a contradiction. Next, let $x \neq y$ then $(x, y) \notin \cap D_X$, so there is a $D \in D_X$ with $(x, y) \notin D$. Choose a symmetric $E$ such that $E \circ E \subset D$ then if $z \in E[x] \cap E[y]$ we have that $(x, z), (z, y) \in E$ implies $(x, y) \in E \circ E \subset D$ which is impossible so $E[x] \cap E[y] = \emptyset$. Clearly the Hausdorff property implies the $T_0$ property for closure spaces. Finally we take a uniformly continuous function $f : \mathbb{R}_2 \to X$ and suppose it is not continuous. Then for $x = f(0), y = f(1)$ and by the $T_0$ property of the underlying closure space we have, without loss of generality, a $D \in D_X$ such that $y \notin D[x]$, hence $(x, y) \notin D$ which implies that $(0, 1) \notin (f \times f)^{-1}(D)$. This contradicts the fact that $f$ was uniformly continuous, so $f$ must be constant. \qed

Definition 5.2.32. Any UPNear-space satisfying one of the above equivalent conditions will be called a Hausdorff uniform pre-nearness space. The full subcategory of UPNear given by these spaces will be written as UPNear$_0$. \bbox

Proposition 5.2.33. A UPNear-space $X$ is Hausdorff if and only if the Unif-bicoreflection $X^u$ is Hausdorff.

Proof. Because $D_X \subset D_X^u$ we have $\cap D_X = \Delta \Rightarrow \cap D_X^u = \Delta$. Conversely, suppose $\cap D_X^u = \Delta$ and take $(x, y) \in \cap D_X \Rightarrow (x, y) \in \cap D_X^u = \Delta$ by definition of $D_X^u$, so $x = y$ and $\cap D_X = \Delta$. \qed

We will now turn to the question of which closure spaces are induced by a uniform pre-nearness space. First we introduce the concept of completely regularity for closure spaces by analogy to topological spaces.

Definition 5.2.34. A closure space $X$ is called a completely regular closure space if and only if for each $x \in X$ and for each closed subset $A$ of $X$, which does not contain $x$, there is a Cl-morphism $f : \overline{X} \to [0, 1]$ with $f(x) = 0$ and $f(A) \subset \{1\}$ (where $[0, 1]$ carries the closure induced by the usual topology). We will write CRegCl for the full subcategory of Cl given by the completely regular closure spaces. \bbox
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For the next theorem we will write $C(X, \mathbb{R})$ for the collection of all $\text{Cl}$-morphisms from $X$ to $\mathbb{R}$ with the usual topology. In the same way we use $C(X, [0, 1])$.

**Theorem 5.2.35.** For a closure space $X$ the following are equivalent:

1. $X$ is completely regular.
2. $(f : X \to [0, 1])_{f \in C(X, [0, 1])}$ is initial.
3. $(f : X \to \mathbb{R})_{f \in C(X, \mathbb{R})}$ is initial.
4. There is a uniform pre-nearness space $Y$ with underlying set $X$ such that $ CY = X $.

**Proof.** We prove the following implications:

(1)$\Rightarrow$(2): Let $C_{in}$ be the initial structure in $\text{Cl}$ given by the source from (2). It is obvious that $C_{in} \subset C_{X}$. Next we choose $x \in X$ and $V$ open neighborhood of $x$, by the completely regularity there is an $f : X \to [0, 1]$ with $f(x) = 0$ and $f(V) \subset \{1\}$ so $f^{-1}([0, 1/2]) \subset V$ is an open neighborhood of $x$ in $C_{in}$ and so is $V$, thus $C_{X} \subset C_{in}$.

(2)$\Rightarrow$(3): Holds because the initiality is kept if one adds more $\text{Cl}$-morphisms to the source.

(3)$\Rightarrow$(4): Let $U_\mathbb{R}$ be the usual uniformity on $\mathbb{R}$. Let $Y$ be the initial lift in $\text{UPNear}$ of the respect to the structured source $(f : X \to (\mathbb{R}, U_\mathbb{R}))_{f \in C(X, \mathbb{R})}$ By a previous result we have that $(f : CY \to \mathbb{R})_{f \in C(Y, \mathbb{R})}$ is also initial, thus $ CY = X $.

(4)$\Rightarrow$(1): Choose a uniform pre-nearness space $Y$ on the set $X$ such that $ CY = X $. Let $a \in X$ and $V$ an open set containing $a$, then there is an $r > 0$ and a $\varphi \in \Psi_X$ for which $\varphi^{-1}([0, r]) \subset V$. We now define $g(x) = \min\{1, 1/r \cdot \varphi_a(x)\}$. One can easily see that $g : X \to [0, 1]$ is a $\text{Cl}$-morphism. Furthermore $g(a) = 0$ and $g(x) = 1$ if and only if $x \notin V$. We can conclude that $X$ is completely regular.

**Corollary 5.2.36.** The functor $ C : \text{UPNear} \to \text{Cl} $ restricts to a functor $ C : \text{UPNear} \to \text{CRegCl} $. 

\(\square\)
As a special case of completely regular closure spaces we give the example of zero-dimensional closure spaces.

**Proposition 5.2.37.** A zero-dimensional closure space is completely regular.

**Proof.** Let $X$ be any zero-dimensional closure space. To see that it is completely regular, take a closed set $A$ and a point $x \not\in A$ in $X$. Since $X$ is zero-dimensional there is a clopen set $B$ such that $A \subset B$ and $x \not\in B$. The CI-morphism $I_B : X \to [0, 1]$ separates $A$ and $x$. 

From this it follows that there always exists a uniform pre-nearness structure compatible with a given zero-dimensional closure space. In the next chapter we will take a closer look at such uniform pre-nearness structures.

### 5.3 Representation of UPNear

In [53] an adjunction is described between the category $\textbf{PNear}$ of pre-nearness spaces and a category $\textbf{PGrL}$ of completely distributive lattices, called pre-grill-lattices. It is shown that this adjunction reduces to an adjunction between $\textbf{Near}$ and a certain subcategory of $\textbf{PGrL}$ consisting of so-called grill-lattices. In [54] the same author describes the restriction of the latter adjunction to the category of uniform nearness spaces and finds an adjunction between $\textbf{Unif}$ and a suitable subcategory of uniform grill-lattices. In this paragraph we will introduce the needed concepts in order to describe the adjunction which arises when one restricts the adjunct situation between $\textbf{PNear}$ and $\textbf{PGrL}$ to the category $\textbf{UPNear}$. For more details on preliminaries we refer the reader to [53] and [54].

**Definition 5.3.1.** For a set $X$ we define the **scale** of $X$ as the set $\sigma(X)$ of all stacks on $X$. We endow the scale with an order relation $\leq$ where $\mathcal{A} \leq \mathcal{B}$ if and only if $\mathcal{B} \subset \mathcal{A}$. Furthermore we write $P(X) = \{\hat{x} | x \in X\}$, where $\hat{x} = \text{stack } \{\{x\}\}$. 

We recall that a complete lattice $\mathcal{L}$ is called completely distributive if arbitrary suprema distribute over arbitrary infima and vice-versa. For a completely distributive lattice $\mathcal{L}$ and $\mathcal{A} \subset P(\mathcal{L})$ one defines $\limsup A = \land\{\lor A | A \in \mathcal{A}\}$ and $\liminf A = \lor\{\land A | A \in \mathcal{A}\}$. 

\[ \begin{array}{c}
\end{array} \]
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**Definition 5.3.2.** A base-lattice is a pair $(\mathcal{L}, B)$ consisting of a completely distributive lattice $\mathcal{L}$ and a subset $B$ of $\mathcal{L}$ such that every $a \in \mathcal{L}$ can be written as $\limsup A$ for a suitable $A \subseteq \mathcal{P}(B)$, i.e. $B$ is a $\limsup$-base of $\mathcal{L}$. In order to keep the notations from getting too technical we shall sometimes write a base-lattice as $\mathcal{L}$ and its base as $B_{\mathcal{L}}$. A morphisms between two base-lattices $\mathcal{L}$ and $\mathcal{M}$ is $\lor, \land$-preserving map $f: \mathcal{L} \rightarrow \mathcal{M}$ such that $f(B_{\mathcal{L}}) \subseteq B_{\mathcal{M}}$. The base-lattices with their morphisms form a category $\text{BL}$. ▶

**Remark 5.3.3.** For any set $X$ the scale $(\sigma(X), \leq)$ is a completely distributive lattice where infima are unions and suprema are intersections. In fact it is a base-lattice with base given by $\mathcal{P}(X)$ since any stack $A = \land\{\lor\{x| x \in A\}| A \in \mathcal{A}\}$. We will write the base-lattice $(\sigma(X), \mathcal{P}(X))$ associated with a set $X$ as $SX$. In [53] this base-lattice is used to construct an adjunct situation between $\text{BL}$ and Set. ▶

Following [53] and [54] we will construct a representation of $\text{PNear}$ and $\text{UPNear}$ using the set of near collections $\xi$.

**Definition 5.3.4.** For a pre-nearness space $\mathcal{X} = (X, \xi)$ one defines the tribe of $\mathcal{X}$ as $\tau(\xi) = \xi \cap \sigma(X)$, i.e. $\tau(\xi)$ is the collection of all near stacks of $\mathcal{X}$. ▶

**Definition 5.3.5.** A pre-grill-lattice is a pair $(\mathcal{L}, G)$ consisting of a base-lattice $\mathcal{L}$ and a proper subset $G$ of $\mathcal{L}$ ($\emptyset \neq G \neq \mathcal{L}$) containing the base $B_{\mathcal{L}}$ such that $G$ is an upperset, i.e. $a \in G, b \leq a$ implies $b \in G$. A morphisms between two pre-grill-lattices $(\mathcal{L}, G)$ and $(\mathcal{M}, H)$ is a $\text{BL}$-morphism $f: \mathcal{L} \rightarrow \mathcal{M}$ such that $f(G) \subseteq H$. Together with these morphisms one obtains the category $\text{PGrL}$ of pre-grill-lattices. ▶

With these definitions we can now formulate the main result of [53] as follows.

**Theorem 5.3.6.** The functors

$G: \text{PNear} \rightarrow \text{PGrL}$

$(X, \xi) \mapsto (SX, \tau(\xi))$

$(f: X \rightarrow Y) \mapsto (G(f): SX \rightarrow SY)$

(where $G(f)(A) = \{E \subseteq Y| f^{-1}(E) \in A\}$)

and

$M: \text{PGrL} \rightarrow \text{PNear}$

$(\mathcal{L}, G) \mapsto (B_{\mathcal{L}}, \xi_G)$

$f \mapsto f|_{B_{\mathcal{L}}}$
5. Uniform pre-nearness spaces

(where $\xi_G = \{ A \subset \mathcal{P}(B_L) | \limsup A \in G \}$) form an adjoint situation with the natural transformations given by:

$$\eta_X : X \rightarrow MG(X) : x \mapsto \hat{x}$$

$$\epsilon_{(L,G)} : GM(L,G) \rightarrow (L,G) : A \mapsto \limsup A$$

Moreover the unit $\eta$ is a natural isomorphism.

In [53] this adjoint situation is studied when it is restricted to the subcategory $\textbf{Near}$ of $\textbf{PNear}$. This yields an adjoint situation between $\textbf{Near}$ and the full subcategory $\textbf{GrL}$ of $\textbf{PGrL}$ whose objects are grill-lattices, i.e. pre-grill-lattices $(L,G)$ such that for $x, y \in L$ one has $x \lor y \in G \iff x \in G$ or $y \in G$. In [54] the adjunction between $\textbf{Near}$ and $\textbf{GrL}$ is restricted to an adjunction between $\textbf{Unif}$ and a suitable subcategory of “uniform” grill-lattices. Using analogous methods we will construct a restriction of the adjoint situation of Theorem 5.3.6 to $\textbf{UPNear}$. From Proposition 5.2.1 we know that a pre-nearness space $(X,\xi)$ is uniform if for every $A \not\in \xi$ there is a $B \not\in \xi$ such that $\{\text{coSt}(B,B) | B \in \mathcal{B}\} <_{co} A$.

In order to find an analogue of costar-co-refinement for pre-grill-lattices we introduce the following notations and lemma’s from [54].

Let $L$ be a base-lattice we write $J(L) = \{ \lor A | A \subset B_L \}$ and for $E \subset L$ we define the downset of $E$, $\downarrow E$ as $\{ a \in L | \exists e \in E : a \leq e \}$. In fact $J(L)$ is a complete lattice with $\lor J(L) = \lor L$ and $\land J(L) = \land L \land E \cap B_L$, for any $E \subset J(L)$.

Let $e \in E \subset J(L)$, we write $t(e,E) = \land J(L) \{ x \in E \} \downarrow \{ e, x \} \cap B_L \neq B_L$ and $t(E) = \{ t(e,E) | e \in E \}$. With these notations a “strictly above”-relation $\gg$ is defined as follows in [54]. For $E, C \subset J(L)$:

$$E \gg C \iff \forall e \in E : \exists c \in C : c \leq t(e,E)$$

We also define the following two maps:

$$\text{Sup} : \mathcal{P}^2(B_L) \rightarrow \mathcal{P}(J(L)) : A \mapsto \{ \lor A | A \in A \}$$

$$\Delta : \mathcal{P}(J(L)) \rightarrow \mathcal{P}^2(B_L) : E \mapsto \{ \partial e | e \in E \}$$

where $\partial e = \downarrow e \cap B_L$. The following lemma is proven in [54].

**Lemma 5.3.7.** For any $E \subset J(L)$ and $A \in \mathcal{P}^2(B_L)$ one has

$$E \gg \text{Sup} A \Rightarrow \{ \text{coSt}(\partial e, \Delta E) | \partial e \in \Delta E \} <_{co} A$$

$$\{ \text{coSt}(A,A) | A \in A \} <_{co} \Delta E \Rightarrow \text{Sup} A \gg E$$
In this way the $\gg$ relation is the counterpart for base-lattices of the costar-co-refinement relation. Hence we can formulate the concept of uniform pre-grill-lattice.

**Definition 5.3.8.** A uniform pre-grill-lattice is a pre-grill-lattice $(\mathcal{L}, G)$ such that for every $C \subset J(\mathcal{L})$ with $\wedge C \notin G$ there exists an $E \subset J(\mathcal{L})$ for which $\wedge E \notin G$ such that $E \gg C$. The full subcategory of $\text{PGrL}$ given by these uniform object will be called $\text{UPGrL}$.

Thus we obtain a restriction of the adjoint situation of Theorem 5.3.6.

**Theorem 5.3.9.** 1. A pre-grill-lattice $(\mathcal{L}, G)$ is uniform if and only if $M(\mathcal{L}, G)$ is a uniform pre-nearness space.

2. A pre-nearness space $(X, \xi)$ is uniform if and only if $G(X, \xi)$ is a uniform pre-grill-lattice.

3. The functors from Theorem 5.3.6 restrict as follows

$$G : \text{UPNear} \to \text{UPGrL}$$

$$M : \text{UPGrL} \to \text{UPNear}$$

and form an adjoint situation with the natural transformations given by restricting $\eta$ and $\epsilon$, moreover $\eta$ is a natural isomorphism.

**Proof.** 1. Let $(\mathcal{L}, G)$ be a uniform pre-grill-lattice, then $M(\mathcal{L}, G) = (\mathcal{B}_L, \xi_G)$. Let $A \in \mathcal{P}^2(\mathcal{B}_L)$ such that $A \notin \xi_G$, i.e. $\limsup A \notin G$. Since $\text{Sup} A \subset J(\mathcal{L})$ and $\wedge \text{Sup} A = \limsup A \notin G$ there exists a subset $E$ of $J(\mathcal{L})$ with $\wedge E \notin G$ and $E \gg \text{Sup} A$. Hence for $B = \Delta E$ we obtain $\{\text{coSt}(B, B)|B \in B\} <_{\text{co}} A$. We also have $\limsup B = \limsup \Delta E = \wedge \{\vee \partial e|\partial e \in \Delta E\} = \wedge E \notin G$, hence $B \notin \xi_G$. Therefore $M(\mathcal{L}, G)$ is a uniform pre-nearness space.

Conversely, suppose $M(\mathcal{L}, G)$ is a uniform pre-nearness space and $C$ is a subset of $J(\mathcal{L})$ such that $\wedge C \notin G$, then $\limsup \Delta C = \wedge \{\vee \partial e|\partial e \in \Delta C\} = \wedge C \notin G$, hence $A = \Delta C \notin \xi_G$. In this case there is a $B \in \mathcal{P}^2(\mathcal{B}_L)$ such that $B \notin \xi_G$ and $\{\text{coSt}(B, B)|B \in \mathcal{B}\} <_{\text{co}} A$. Writing $E$ for $\text{Sup} B \supset J(\mathcal{L})$ one has $E \gg C$ and since $\limsup B \notin G$ also $\wedge E \notin G$. Thus $(\mathcal{L}, G)$ is a uniform pre-grill-lattice.

2. By Theorem 5.3.6 we know that $\eta_X : X \to M\Gamma_X$ is an isomorphism in $\text{PNear}$. Hence $X$ is uniform pre-nearness space if and only if the same holds for $M\Gamma_X$, which by the preceding proof is true if and only if $\Gamma_X$ is a uniform pre-grill-lattice.
3. This is an immediate result from the above in combination with Theorem 5.3.6.

This adjoint situation does not reduce to an equivalence is shown by the next example.

**Example 5.3.10.** [A uniform pre-grill-lattice such that $\epsilon$ is not an isomorphism.]

Let $\mathcal{L}$ be the lattice shown in Figure 5.1, together with the base $B_{\mathcal{L}} = \{a, b\}$ it is a base-lattice. In order to make a pre-grill-lattice out of $\mathcal{L}$ one must have an upperset $G$ containing $B_{\mathcal{L}}$, hence $G = \{a, b, I\}$. The corresponding uniform pre-nearness space $(B_{\mathcal{L}}, \xi_G)$ is given by the structure

$$\xi_G = \{\{a, b\}, \{\{a\}\}, \{\{b\}\}, \hat{a}, \hat{b}, \emptyset\}$$

which is in fact the discrete $\text{PNear}$-object on $\{a, b\}$. Its tribe is $\tau(\xi_G) = \{\{a, b\}, \hat{a}, \hat{b}, \emptyset\}$ and its scale is given as in Figure 5.2, where

$$1 = \mathcal{P}(\{a, b\}), 2 = \hat{a} \cup \hat{b}, 3 = \hat{a}, 4 = \check{b}, 5 = \hat{a} \cap \check{b}, 6 = \emptyset$$

Since $(B_{\mathcal{L}}, \xi_G)$ is discrete both $(\mathcal{L}, G)$ and $(SB_{\mathcal{L}}, \tau(\xi_G)$ are uniform pre-grill-lattices. However it is obvious that $\epsilon_{(\mathcal{L}, G)}$ is not bijective, hence it cannot be an isomorphism.
Figure 5.2: The scale of $(B_{\mathcal{L}}, \xi_G)$. 
Chapter 6

Non-Archimedean uniform pre-nearness spaces

As is explained in the introductory part of the previous chapter, our final aim is to develop a completion theory which somehow incorporates the zero-dimensional closure spaces, since these spaces have important applications (convex analysis, quantum logic, ...). We have already seen that each 0Cl-space is uniformizable (Proposition 5.2.37). In the uniform case non-Archimedean uniform spaces were introduced by A.F. Monna in [36] as uniform spaces having a base consisting of only equivalence relation. These spaces have turned out to be the uniform counterparts of zero-dimensional topological spaces. We introduce non-Archimedean spaces in a similar way.

The Hausdorff non-Archimedean uniform spaces and uniformly continuous maps form a category which does have well-behaved complete objects [13]. We shall prove that the same holds for our Hausdorff non-Archimedean spaces. Moreover it will be shown that the completion of a Hausdorff non-Archimedean space reduces to the usual completion when the space is a uniform space. The results obtained in this chapter have been established in collaboration with E. Lowen-Colebunders and were published in [21].

6.1 The category NA

Definition 6.1.1. For any pre-nearness space \( X \) we will write \( \nu_X \) for the set \( \{ U \in \mu_X | U \text{ is a partition} \} \). A non-Archimedean pre-nearness space or shortly a
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non-Archimedean space is a pre-nearness space \( X \) such that \( \nu_X \) generates \( \mu_X \), i.e. \( \mu_X = \operatorname{stack}_X \nu_X \). The full subcategory of \( \mathbf{PNear} \) given by the non-Archimedean spaces will be denoted by \( \mathbf{NA} \).

In this way we closely follow B. Banaschewski, who stated in [13] that non-Archimedean structures “belong to the subfield of general topology that can be described by means of equivalence relations”.

Note that every partition star-refines itself, hence a non-Archimedean space is also a uniform pre-nearness space. This implies the following proposition.

**Proposition 6.1.2.** Let \( X \) be a uniform pre-nearness space and let \( \mathcal{E}_X = \{ E \in \mathcal{D}_X | E \text{ is an equivalence relation} \} \). Then \( X \) is a non-Archimedean space if and only if \( \mathcal{E}_X \) is a base for \( \mathcal{D}_X \), i.e. \( \mathcal{D}_X = \operatorname{stack} \mathcal{E}_X \).

We have shown that one can also describe a uniform pre-nearness space by means of a family of pseudometrics. In the case of a non-Archimedean space this family is generated by ultra-pseudometrics.

**Definition 6.1.3.** A pseudometric \( \psi \) on a set \( X \) is called an ultra-pseudometric if it satisfies the strong triangular inequality:

\[
\psi(x, y) \leq \max \{\psi(x, z), \psi(z, y)\}
\]

for all \( x, y, z \in X \).

**Proposition 6.1.4.** Let \( \psi \) be a pseudometric on a set \( X \). Then \( \psi \) is an ultra-pseudometric if and only if for every \( \epsilon > 0 \), \( E = \psi^{-1}([0, \epsilon]) \) is an equivalence relation.

**Proof.** Obviously \( E = \psi^{-1}([0, \epsilon]) \) is symmetric and reflexive. To show transitivity, suppose \((x, y), (y, z) \in E \). Then both \( \psi(x, y) \) and \( \psi(y, z) \) are less than \( \epsilon \), hence \( \psi(x, z) \leq \max \{\psi(x, y), \psi(y, z)\} < \epsilon \). Thus we obtain \((x, z) \in E \).

Conversely, if for some \( x, y, z \in X \) one has \( \psi(x, y) > \max \{\psi(x, z), \psi(z, y)\} \) then there exists an \( \epsilon > 0 \) for which \( \psi(x, y) > \epsilon > \max \{\psi(x, z), \psi(z, y)\} \). Hence \((x, z), (z, y) \in \psi^{-1}([0, \epsilon]) = E \). Since \( E \) is an equivalence relation this implies \((x, y) \in E = \psi^{-1}([0, \epsilon]) \) which contradicts our choice of \( \epsilon \). Therefore \( \psi \) is an ultra-pseudometric.

**Proposition 6.1.5.** Let \( E \subset X \times X \) then \( \psi(x, y) = \begin{cases} 0 & (x, y) \in E \\ 1 & (x, y) \notin E \end{cases} \) defines an ultra-pseudometric if and only if \( E \) is an equivalence relation.
Proof. One implication is a direct consequence of the previous proposition. For the other one observes that $\psi$ is a pseudometric and, for each $\epsilon > 0$, $\psi^{-1}([0, \epsilon])$ is an equivalence relation, hence by the previous proposition $\psi$ is an ultra-pseudometric.

We can now reformulate the definition of a non-Archimedean space in terms of pseudometrics.

**Proposition 6.1.6.** For a uniform pre-nearness space $X$ we write $\Phi_X$ for the set of all ultra-pseudometrics on $X$ contained in $\Psi_X$. Then $X$ is a non-Archimedean space if and only if for every $\psi \in \Psi_X$ we have

$$\forall \epsilon > 0 : \exists \varphi \in \Phi_X: \exists \delta > 0 : \varphi(x, y) < \delta \Rightarrow \psi(x, y) < \epsilon$$

We will use the following notations. A non-Archimedean space will be written as $\underline{X}$ and we shall use $\nu_X$, $\mathcal{E}_X$ and $\Phi_X$ to refer to the subsets of $\mu_X$, $\mathcal{D}_X$ and $\Psi_X$ described above. We will write $[x]_P$ as well as $E[x]$ for the equivalence class of a point $x \in X$ with respect to a partition $P \in \nu_X$ or an equivalence relation $E \in \mathcal{E}_X$.

Thus, if $\underline{X}^u$ is the $\mathsf{Unif}$-bicoreflection of a non-Archimedean space $\underline{X}$, one obtains that $\mathcal{E}_{\underline{X}^u}$ is the saturation of $\mathcal{E}_X$ for finite intersections, hence $\underline{X}^u$ is a non-Archimedean uniform space [13], [36], [49], [50]. Therefore the full subcategory $\mathsf{NAUnif}$ of $\mathsf{Unif}$ consisting of the non-Archimedean uniform spaces is a bicoreflective subcategory of $\mathsf{NA}$.

We now turn to some other interesting features of $\mathsf{NA}$ which we shall need further on.

**Proposition 6.1.7.** $\mathsf{NA}$ is a bireflective subcategory of $\mathsf{UPNear}$.

Proof. As in the case of a non-Archimedean uniform space, the bireflection of an uniform pre-nearness space $\underline{X}$ is given by the map $1_X : \underline{X} \to \underline{X}^{na}$, where $\mathcal{D}_{\underline{X}^{na}} = \text{stack} \ \mathcal{E}_X$. 

As a bireflective subcategory of $\mathsf{UPNear}$, $\mathsf{NA}$ is a well-libred topological construct (with the usual forgetful functor $U$). The $\mathsf{NA}$-structure which gives an initial lift for a structured source $(f_i : X \to UX_i)_{i \in I}$, $X_i \in [\mathsf{NA}]$ is given by the base $\{(f_i \times f_i)^{-1}(E_i)| i \in I, E_i \in \mathcal{E}_X \}$ for $\mathcal{D}_X$. As a topological category $\mathsf{NA}$
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has $T_0$-objects. We write $\text{NA}_0$ for the subconstruct consisting of the $T_0$-objects of $\text{NA}$. A $\text{NA}_0$-object is a non-Archimedean space $X$ such that, for any two different points $x$ and $y$ in $X$, there is an equivalence relation $E \in \mathcal{E}_X$ such that $E[x] \neq E[y]$, i.e. the associated uniform pre-nearness space is Hausdorff. As for pre-nearness spaces one has a closure space associated with a non-Archimedean space. The following proposition shows that this leads to a restriction of the functor $C : \text{UPNear} \to \text{CRegCl}$ to $C : \text{NA} \to \text{0Cl}$.

**Proposition 6.1.8.** Let $X$ be a non-Archimedean space then for each $x \in X$ the set $\{E[x] | E \in \mathcal{E}_X\}$ is a base for a neighborhood stack of $x$. Moreover the closure structure $C(X)$ associated with these neighborhood stacks is zero-dimensional.

**Proof.** It suffices to remark that the collection $\{E[x] | E \in \mathcal{E}_X\}$ consists of clopen sets.

We also have a converse to this proposition.

**Proposition 6.1.9.** Let $X$ be a zero-dimensional closure space. Then the collection $\nu = \{\{A, X \setminus A\} | A \text{ clopen in } X\}$ generates a non-Archimedean space $X_\nu$ on $X$. Moreover it’s underlying closure is the given one.

**Proof.** We prove that both closure structures have the same neighborhood stacks for $x \in X$. For every clopen neighborhood $V$ of $x$ in $X_\nu$ we have that $\{V, X \setminus V\}$ is a member of $\nu$, so by the previous proposition $V$ is a clopen neighborhood of $x$ in $C(X_\nu)$. Conversely if $V$ is a basic neighborhood of $x$ in the $C(X_\nu)$ then $\{V, X \setminus V\}$ a partition $\mathcal{U}$ of clopen sets of $X$, hence $V = [x]_\mathcal{U}$ is a neighborhood of $x$ in $X_\nu$.

6.2 Representation of NA

$\text{NA}$ is a subcategory of $\text{UPNear}$, as such it is a valid question whether the adjunction from Theorem 5.3.9 can be restricted to $\text{NA}$. In order to show that this is indeed possible we will give a characterization of the non-Archimedean spaces using near collections. We define the concept of partitioning collection as follows.

**Definition 6.2.1.** For any set $X$ a **partitioning collection** is a collection $\mathcal{A}$ of subsets of $X$ such that $\cap \mathcal{A} = \emptyset$ and for any two different $A_1, A_2 \in \mathcal{A}$ one has that $A_1 \cup A_2 = X$. □
Proposition 6.2.2. Let \( \mathfrak{X} \) be a \( \text{PNear} \)-object. The following are equivalent:

1. \( \mathfrak{X} \) is a non-Archimedean space, i.e. for every \( U \in \mu_{\mathfrak{X}} \) there is a partition \( V \in \mu_{\mathfrak{X}} \) for which \( V < U \).

2. For every \( A \in \mathcal{P}_2(\mathfrak{X}) \) with \( A \not\in \xi_{\mathfrak{X}} \) there exists a partitioning collection \( B \in \xi_{\mathfrak{X}} \) such that \( B < \co A \).

Proof. Let \( A \in \mathcal{P}_2(\mathfrak{X}) \) with \( A \not\in \xi_{\mathfrak{X}} \), then \( \{ \mathfrak{X} \setminus A | A \in A \} \in \mu_{\mathfrak{X}} \), hence it is refined by a partition \( \mathcal{V} \). Let \( B = \{ \mathfrak{X} \setminus V | V \in \mathcal{V} \} \not\in \xi_{\mathfrak{X}} \). Since \( B \not\in \xi_{\mathfrak{X}} \) we have that \( \cap \mathcal{B} \) is empty. Moreover if \( B_1, B_2 \in \mathcal{B} \) then \( B_1 \cup B_2 = \mathfrak{X} \setminus (V_1 \cap V_2) \) for some \( V_1, V_2 \in \mathcal{V} \). Because \( \mathcal{V} \) is a partition this implies that \( \mathcal{B} \) is a partitioning collection such that \( \mathcal{B} < \co A \). Conversely, suppose \( U \in \mu_{\mathfrak{X}} \) then this implies that \( A = \{ \mathfrak{X} \setminus U | U \in U \} \not\in \xi_{\mathfrak{X}} \), therefore there exists a partitioning collection \( \mathcal{B} \not\in \xi_{\mathfrak{X}} \) which corefines \( A \). Let \( \mathcal{V} = \{ \mathfrak{X} \setminus B | B \in \mathcal{B} \} \in \mu_{\mathfrak{X}} \). Since \( \mathcal{B} \) is a partitioning collection \( \mathcal{V} \) is a partition. Moreover we have that \( \mathcal{V} < U \). ❄

This proposition characterizes the non-Archimedean spaces using near collections. We now turn to the problem of defining “non-Archimedean” pre-grill-lattices. We recall that, for a base lattice \( \mathcal{L} \), \( J(\mathcal{L}) = \{ \lor A | A \in \mathcal{L} \} \).

Definition 6.2.3. Let \( \mathcal{L} \) be a base lattice. A subset \( C \) of \( \mathcal{L} \) is a partitioning set if \( \cap \Delta C = \emptyset \) and \( \partial c_1, \partial c_2 \in C \) implies \( \partial c_1 \cup \partial c_2 = B_{\mathcal{L}} \).

Lemma 6.2.4. Let \( \mathcal{L} \) be a base lattice and \( C \) a partitioning set of \( \mathcal{L} \), then \( C \gg C \).

Proof. Let \( c \in C \). Since \( t(c, C) = \land J(\mathcal{L}) \{ x \in C | \downarrow \{ c, x \} \cap B_{\mathcal{L}} \neq B_{\mathcal{L}} \} \) and \( \downarrow \{ c, x \} \cap B_{\mathcal{L}} = \partial c \cup \partial x = B_{\mathcal{L}} \) for \( x \in C \) we obtain that \( t(c, C) = \land J(\mathcal{L}) \emptyset = \top J(\mathcal{L}) \). Therefore clearly \( C \gg C \). ❄

Definition 6.2.5. Let \( (\mathcal{L}, G) \) be a pre-grill lattice, it is called a non-Archimedean pre-grill-lattice if for each \( C \in J(\mathcal{L}) \) with \( \land C \not\in G \) one has a partitioning set \( E \) with \( \land E \not\in G \) such that \( \forall e \in E : \exists e \in C : e \leq e \).

Using the preceding lemma one sees that any non-Archimedean pre-grill-lattice is in fact a uniform one, hence we write \( \text{NAPGrL} \) for the full subcategory of \( \text{UPGrL} \) given by the non-Archimedean pre-grill-lattices. The following theorem gives a restriction of the adjunction of Theorem 5.3.9 to the non-Archimedean spaces.
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**Theorem 6.2.6.** 1. A pre-grill-lattice \((\mathcal{L}, G)\) is non-Archimedean if and only if \(M(\mathcal{L}, G)\) is a non-Archimedean space.

2. A pre-nearness space \((X, \xi)\) is non-Archimedean if and only if \(G(X, \xi)\) is a non-Archimedean pre-grill-lattice.

3. The functors from Theorem 5.3.9 restrict as follows

\[
G : NA \rightarrow NAPGrL
\]

\[
M : NAPGrL \rightarrow NA
\]

and form an adjoint situation with the natural transformations given by restricting \(\eta\) and \(\epsilon\), moreover \(\eta\) is a natural isomorphism.

**Proof.** 1. Let \((\mathcal{L}, G)\) be a pre-grill-lattice, then \(M(\mathcal{L}, G) = (B_\mathcal{L}, \xi_G)\). Let \(A \in \mathcal{P}^2(B_\mathcal{L})\) such that \(A \notin \xi_G\), i.e. \(\limsup A \notin G\). Since \(\text{Sup} A \subset J(\mathcal{L})\) and \(\wedge \text{Sup} A = \limsup A \notin G\) there exists a partitioning set \(E \subset J(\mathcal{L})\) with \(\wedge E \notin G\) and \(\forall e \in E : \exists A \in A : \forall A \leq e\). Writing \(B\) for \(\Delta E\) we find that \(B <_{\text{co}} A\) and that \(B \notin \xi_G\). Finally \(\cap B = \cap \Delta E = \emptyset\) and for \(B_1, B_2 \in B\) we know that \(B_1 \cup B_2 = \partial e_1 \cup \partial e_2 = B_\mathcal{L}\), hence \(E\) is a partitioning collection. Thus \(M(\mathcal{L}, G)\) is a non-Archimedean space.

Conversely, suppose \(M(\mathcal{L}, G)\) is a non-Archimedean space and \(C\) is a subset of \(J(\mathcal{L})\) such that \(\wedge C \notin G\), then \(\limsup \Delta C = \wedge \{\forall e \in \Delta C\} = \wedge C \notin G\), hence \(A = \Delta C \notin \xi_G\). In this case there is a partitioning collection \(B \in \mathcal{P}^2(B_\mathcal{L})\) such that \(B \notin \xi_G\) and \(B <_{\text{co}} A\). Let \(E\) be \(\text{Sup} B \supset J(\mathcal{L})\), one has for any \(B \in B\) a \(\partial c \in A = \Delta C\) such that \(\partial c \leq \forall B\). Moreover since \(x \in \cap \Delta E = \cap \{\forall \partial B | B \in B\}\) would imply that \(x \leq \limsup B\), which is impossible because \(B \notin \xi_G\), one finds that \(\cap \Delta E = \emptyset\). Also, for any two \(B_1, B_2 \in B\) we know that \(B_1 \cup B_2 = B_\mathcal{L}\), hence \(\partial (\forall B_1) \cup \partial (\forall B_2) = B_\mathcal{L}\), hence \(E\) is a partitioning set. Thus \((\mathcal{L}, G)\) is non-Archimedean.

2. By Theorem 5.3.9 we know that \(\eta_X : X \rightarrow MG_X\) is an isomorphism in \(\text{PNear}\). Hence \(X\) is a non-Archimedean space if and only if \(MG_X\) is a non-Archimedean space, which by the preceding proof is true if and only if \(G_X\) is a non-Archimedean pre-grill-lattice.

3. This is an immediate result from the above in combination with Theorem 5.3.9.

\(\square\)
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Example 6.2.7. [A non-Archimedean pre-grill-lattice.]
The discrete $\text{UPnear}$-objects are non-Archimedean spaces, hence the pre-nearness space obtained in Example 5.3.10 is non-Archimedean, therefore both pre-grill-lattices from that example are non-Archimedean pre-grill-lattices. ◀

6.3 Completion of $\mathbf{NA}_0$-spaces

6.3.1 The firm $\mathcal{U}_{\mathbf{NA}_0}$-reflective subcategory of $\mathbf{NA}_0$

Since we would like to apply Theorem 4.1.6 to $\mathbf{NA}$, we start by proving that $\mathbf{NA}_0$ is $\text{Emb } \mathbf{NA}_0$-cogenerated by the discrete $\mathbf{NA}_0$-spaces.

Proposition 6.3.1. Let $X$ be a Hausdorff non-Archimedean space. For each $P \in \nu_X$ we write $D_P$ for the discrete non-Archimedean space on the set $P$. We have that

$$i : X \rightarrow \Pi_{P \in \nu_X} D_P : x \mapsto ([x]_P)_{P \in \nu_X}$$

is an embedding in $\mathbf{NA}_0$.

Proof. Consider the source:

$$(i_P : X \rightarrow D_P : x \mapsto [x]_P)_{P \in \nu_X}$$

Since $(i_P \times i_P)^{-1}(\Delta_P) = E_P$, where $E_P$ denotes the equivalence relation defined by $P$, we know that $\nu_X$ is the initial structure for this source. Because $X$ is Hausdorff we have that the above source is point separating. Hence

$$i : X \rightarrow \Pi_{P \in \nu_X} D_P : x \mapsto ([x]_P)_{P \in \nu_X}$$

is an embedding. □

This means that the class of Hausdorff non-Archimedean spaces coincides exactly with the epireflective hull in $\mathbf{NA}$ of the class of all discrete objects, or using the terminology of [19], $\mathbf{NA}_0$ is $\text{Emb } \mathbf{NA}_0$-cogenerated by the class of all discrete objects, i.e. the Hausdorff non-Archimedean spaces are exactly the subspaces of products of discrete spaces.

To simplify notations we shall write $\Pi_{P \in \nu_X} D_P$ instead of $\Pi_{P \in \nu_X} D_P$. 

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Example 6.3.2. [A Hausdorff non-Archimedean space, which is not a uniform space, embedded in $D_3 \times D_4$.]

Let $X = \{1, 2, 3, 4, 5\}$ and $\overline{X}$ the non-Archimedean space defined by $D_X = \text{stack } \{E_1, E_2\}$, where $E_1$ and $E_2$ are the following two equivalence relations (see Figure 6.1):

$E_1 = \{(3, 1), (3, 2), (2, 1), (2, 2), (1, 1), (3, 3), (5, 5), (4, 4), (2, 3), (1, 3), (1, 2)\}$

$E_2 = \{(2, 2), (1, 1), (3, 3), (5, 5), (4, 4), (4, 5), (5, 4)\}$

The structure of $\overline{X}$ can also be described by the corresponding partitions:

$$P_1 = \{\{1, 2, 3\}, \{4\}, \{5\}\}$$

$$P_2 = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$$

This space is clearly Hausdorff since $E_1 \cap E_2 = \Delta_X$. The associated closure space is the discrete one on $X$. One can easily see that $\overline{X}$ is a subspace of $D_3 \times D_4$.

Since we know by Proposition 6.3.1 that $\text{NA}_0$ is $\text{Emb} \text{NA}_0$-cogenerated by the discrete $\text{NA}_0$-spaces, we can apply Theorem 4.1.6 to find a firm $\mathcal{U}_{\text{NA}_0}$-reflective subcategory of $\text{NA}_0$. Before we do so, we observe the following facts.
Remark 6.3.3. From the results of T. Marny in [34] it follows that \( \mathbf{NA}_0 \) is an extremally epireflective subconstruct of \( \mathbf{NA} \) and as such it is initially structured in the sense of [37], [43]. In particular \( \mathbf{NA}_0 \) is complete and well-powered, it is an (epi, extremal mono) category and an (extremal epi, mono) category [32]. Also from the general setting [37] it follows that the monomorphisms in \( \mathbf{NA}_0 \) are exactly the injective uniformly continuous maps and a morphism in \( \mathbf{NA}_0 \) is an extremal epimorphism if and only if it is a regular epimorphism if and only if it is surjective and final. ▶

In order to give a description of the class of epimorphic embeddings \( \mathcal{U}_{\mathbf{NA}_0} \), we need the regular \( \mathbf{NA}_0 \)-closure. By Proposition 6.3.1 we know that this closure operator is described in the following way. For a non-Archimedean space \( X \) and for any \( M \subset X \), \( x \in \text{reg}_{\mathbf{NA}_0} X(M) \) if and only if for every discrete space \( D \) and for every pair of uniformly continuous maps \( f, g : X \to D \)

\[
f|_M = g|_M \Rightarrow f(x) = g(x)
\]

In order to obtain an explicit description of the regular closure operator we introduce the following closure.

Definition 6.3.4. Let \( X \) be a non-Archimedean space and \( M \subset X \). \( x \in \zeta_X(M) \) if and only if for every two equivalence relations \( E_1, E_2 \in \mathcal{E}_X \), which coincide on \( M \), we have that \( E_1[x] \cap E_2[x] \cap M \neq \emptyset \). ▶

Proposition 6.3.5. For every non-Archimedean space \( X \) and \( M \subset X \):

\[
\zeta_X(M) = \text{reg}_{\mathbf{NA}_0} X(M)
\]

Proof. Let \( X \) be a non-Archimedean space, let \( x \in X \), \( M \subset X \) such that \( x \notin \text{reg}_{\mathbf{NA}_0} X(M) \). There is a discrete object \( D \) and there are uniformly continuous maps \( f, g : X \to D \) for which \( f|_M = g|_M \) and \( f(x) \neq g(x) \). Consider \( E_1 = (f \times f)^{-1}((\Delta_D)), E_2 = (g \times g)^{-1}((\Delta_D)) \). Clearly \( E_1 \) and \( E_2 \) belong to \( \mathcal{E}_X \) but do not satisfy the condition in Definition 6.3.4.

Conversely, if \( x \notin \zeta_X(M) \), choose \( E_1, E_2 \in \mathcal{E}_X \) such that \( E_1 \) and \( E_2 \) coincide on \( M \) and \( E_1[x] \cap E_2[x] \cap M = \emptyset \). Let \( C = \{E_1[m] \cap M | m \in M\} = \{E_2[m] \cap M | m \in M\} \) and \( D = C \cup \{a, b\} \) where \( a, b \notin C \). Write \( D \) for the discrete object on \( D \). We define the following functions:

\[
f : X \to D : y \mapsto \begin{cases} E_1[m] \cap M & \text{if } \exists m \in M : (y, m) \in E_1 \\ a & \text{if } \forall m \in M : (y, m) \notin E_1 \end{cases}
\]
and
\[ g : X \to D : y \mapsto \begin{cases} E_2[m] \cap M & \text{if } \exists m \in M : (y, m) \in E_2 \\ b & \text{if } \forall m \in M : (y, m) \notin E_2 \end{cases} \]

Clearly \( f \) and \( g \) are uniformly continuous, they coincide on \( M \) but \( f(x) \neq g(x) \).

It follows that
\[ \zeta = \{ \xi_X : \mathcal{P}(X) \to \mathcal{P}(X) \}_{X \in \text{NA}} \]
defines a closure operator on \( \text{NA} \).

**Lemma 6.3.6.** Let \( X \) be a non-Archimedean space and \( Y \subset X \) then: for every \( E \in \mathcal{E}_X \) there is an \( E' \in \mathcal{E}_Y \) for which \( E = E' \cap Y \times Y \).

**Proof.** We consider the non-empty collection:
\[ R = \{ E' \in \mathcal{E}_Y | E' \cap Y \times Y \subset E \} \]
Since for every ascending chain \( (E_i)_{i \in I} \) in \( R \) we have that \( \cup_{i \in I} E_i \) is also in \( R \), applying Zorn’s Lemma we get the existence of a maximal element \( E' \) of \( R \). If \( (a, b) \in E \) then we consider:
\[ F' = E' \cup \{(u, v)|(u, a), (b, v) \in E'\} \cup \{(u, v)|(u, b), (a, v) \in E'\} \]
After some tedious verifications we find that \( F' \) is an equivalence relation satisfying \( F' \cap Y \times Y \subset E \). Hence \( F' \in R \). By the maximality of \( E' \) we have \( E' = F' \). Finally, because \( E \) is an equivalence on \( Y \) and \( (a, b) \in E \) were chosen arbitrarily we find that \( E \subset E' \cap Y \times Y \subset E \), hence we have found the equivalence \( E' \) we were looking for.

**Proposition 6.3.7.** \( \zeta \) is hereditary, i.e. for a space \( Y \) and a subspace \( X \) in \( \text{NA} \) and \( M \subset X \subset Y \), we have \( \xi_X(M) = \xi_Y(M) \cap X \).

**Proof.** Clearly \( \xi_X(M) \subset \xi_Y(M) \cap X \). Conversely, if \( y \notin \xi_X(M) \) there are equivalences \( E_1, E_2 \) in \( \mathcal{E}_X \) such that they coincide on \( M \) and \( E_1[y] \cap E_2[y] \cap M = \emptyset \). By Lemma 6.3.6 we have two equivalences \( E'_1, E'_2 \), which coincide on \( X \) and satisfy \( E'_1[y] \cap E'_2[y] \cap M \cap X = \emptyset \). Therefore we also have \( E'_1[y] \cap E'_2[y] \cap M = \emptyset \), thus \( y \notin \xi_X(M) \cap X \). From this it follows that \( \zeta \) is hereditary.

From this closure operator \( \zeta \), we get the following characterization of the \( \text{NA}_0 \)-morphisms.
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Proposition 6.3.8. In \( \mathbf{NA}_0 \) we have:

1. The epimorphisms are exactly the \( \zeta \)-dense uniformly continuous maps.

2. The extremal monomorphisms are the regular monomorphisms and they both coincide with the \( \zeta \)-closed embeddings.

Proof. 1. This follows since \( \zeta \) is the regular closure operator determined by \( \mathbf{NA}_0 \).

2. Using Proposition 4.1.11 and the fact that \( \mathbf{NA}_0 \) is extremally epireflective in \( \mathbf{NA} \), for an \( \mathbf{NA}_0 \) morphism \( f: X \to Y \) the following implications hold. (i) \( f \) \( \zeta \)-closed embedding \( \Rightarrow \) (ii) \( f \) regular monomorphism \( \Rightarrow \) (iii) \( f \) extremal monomorphism. To see that (iii) implies that \( f \) is \( \zeta \) closed, we have to use (weak) hereditariness of \( \zeta \). Let \( M = \zeta_Y(f(X)) \) and \( h: M \to Y \) the associated \( \zeta \)-closed embedding. Then there exists a unique map \( g \) such that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{h} \\
M & \xrightarrow{h} & Y
\end{array}
\]

By the (weak) hereditariness of the \( \zeta \)-closure, \( g \) is \( \zeta \)-dense and so it is an epimorphism in \( \mathbf{NA}_0 \). It follows that \( g \) is an isomorphism and then \( f \) is \( \zeta \)-closed.

\[\square\]

Remark 6.3.9. From the previous characterization of the epimorphisms in \( \mathbf{NA}_0 \) it now follows that \( \mathbf{NA}_0 \) is co-well-powered. It suffices to observe that given an \( \zeta \)-dense map \( f: X \to Y \), with \( X \) fixed and writing \( Z \) for the subspace \( f(X) \) of \( Y \), there is a one to one correspondence between \( \nu_Z \) and \( \nu_Y \). Therefore the cardinality of \( Y \) is uniformly bounded.

Proposition 6.3.10. Every discrete non-Archimedean space is \( \mathcal{U}_{\mathbf{NA}_0} \)-injective.

Proof. Let \( u: X \to Y \) be a \( \zeta \)-dense embedding between two Hausdorff non-Archimedean spaces, and let \( f: X \to D \) be a uniformly continuous function to a discrete space. By the initiality of \( u \) we have an \( E \in \mathcal{E}_Y \) such
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that \((u \times u)^{-1}(E) = (f \times f)^{-1}(\Delta_D)\). For \(y \in Y\) we choose \(x_y \in X\) such that \((y, u(x_y)) \in E\). We define

\[
f^* : Y \to D : y \mapsto f(x_y)
\]

Clearly \(f^*\) is a well-defined uniformly continuous map which is an extension of \(f\) along \(u\).

Now that we know that \(\text{NA}_0\) is a complete, well-powered construct, \(\text{Emb NA}_0\)-cogenerated by \(\text{U NA}_0\)-injective objects, we can apply Theorem 4.1.6.

**Proposition 6.3.11.** \(\text{NA}_0\) admits a firm \(\text{U NA}_0\)-reflective subcategory, given by the epireflective hull \(\varepsilon_{\text{NA}_0}(D)\) in \(\text{NA}_0\) (here \(D\) is the class of all discrete spaces).

Next we present an internal characterization of the objects in the firm \(\text{U NA}_0\)-reflective subconstruct. In order to do so we remark that for a non-Archimedean space \(X\), the refinement relation \(<\) on \(\nu_X\) is an partial order, hence we can formulate the following.

**Definition 6.3.12.** Let \(X\) be a non-Archimedean space. A **choice function** is a map \(f : \nu_X \to \cup\nu_X\) such that for every \(P \in \nu_X\) one has that \(f(P) \in P\). A choice function is order preserving if and only if \(P < Q\) implies \(f(P) \subset f(Q)\).

**Definition 6.3.13.** A **complete Hausdorff non-Archimedean space** is a Hausdorff non-Archimedean space such that for every order preserving choice function \(f\) there is an \(x \in \cap_{P \in \nu_X} f(P)\). The point \(x\) is called a limit point of \(f\) and we will say that \(f\) converges to \(x\). Note that in this case the limit point \(x\) is unique.

The following proposition links this notion of completeness to the firm \(\text{U NA}_0\)-reflective subcategory we described before.

**Proposition 6.3.14.** Let \(X\) be a Hausdorff non-Archimedean space and consider the embedding \(i : X \to \Pi\nu_X\) from Proposition 6.3.1. We have the following equivalence.

\[
z = (z_P)_{P \in \nu_X} \in \varepsilon_{\nu_X}(i(X)) \iff \forall P, Q \in \nu_X : P < Q \Rightarrow z_P \subset z_Q
\]

**Proof.** First let \(z \in \varepsilon_{\nu_X}(i(X))\) and let \(P < Q\), where \(P, Q \in \nu_X\). For \(E_P = (pr_P \times pr_P)^{-1}(\Delta_P)\) and \(E_Q = (pr_Q \times pr_Q)^{-1}(\Delta_Q)\), we know there is an
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$x \in X$ such that $i(x) \in E_P[z] \cap E_Q[z]$. Therefore $[x]_P = z_P \in P$ and $[x]_Q = z_Q \in Q$. Because $P < Q$ there is a $Q \in Q$ for which $z_P \subset Q$. Since $z_Q \in Q$ and both $Q$ and $z_Q$ contain $x$ we have that $z_Q = Q$. Hence $z_P \subset z_Q$.

Conversely, choose $E_1, E_2$ in $\mathcal{E}_{\Pi^\nu_X}$ such that both coincide on $i(X)$. Clearly one can write $E_1 = (pr_P \times pr_P)^{-1}(E_P)$ and $E_2 = (pr_Q \times pr_Q)^{-1}(E_Q)$ where $E_P$ and $E_Q$ are equivalences on $P$ and $Q$ respectively.

For the equivalence relation $E_P$, consider classes $[x]_P, [y]_P \in P$ of points $x$ and $y$ in $X$, we have

$$([x]_P, [y]_P) \in E_P \iff (i(x), i(y)) \in E_1 \iff (i(x), i(y)) \in E_2 \iff ([x]_Q, [y]_Q) \in E_Q$$

Hence $R = \{(y \in X | ([x]_P, [y]_P) \in E_P) | x \in X\}$ is a partition of $X$, such that $P, Q < R$. So by the hypothesis we have $z_P, z_Q \subset z_R$. Thus there is an $x \in X$ for which $(i(x), z) \in E_1$ and $(i(x), z) \in E_2$. Finally we have $z \in \zeta_{\Pi^\nu_X}(i(X))$.  

**Theorem 6.3.15.** Let $X$ be a non-Archimedean space. The following are equivalent:

1. $X$ is complete.

2. $X$ is a $\zeta$-closed subspace of $\Pi^\nu_X$.

3. $X$ is $U_{\mathbf{NA}_0}$-injective.

**Proof.** We prove the following implications:

1$\Rightarrow$2: Suppose $X$ is complete. We prove that $i(X)$ is $\zeta$-closed in $\Pi^\nu_X$. Let $z = (z_P)_{P \in \nu_X} \in \zeta_{\Pi^\nu_X}(i(X))$, by Proposition 6.3.14 we know that $f : \nu_X \to \cup \nu_X : P \mapsto z_P$ is an order preserving choice function. Clearly the completeness of $X$ guarantees the existence of $x \in X$ such that $z = i(x)$.

2$\Rightarrow$3: Let $X$ be a $\zeta$-closed subspace of $\Pi^\nu_X$. Since $\mathbf{NA}_0$ is complete, well-powered and co-well-powered, applying 37.6 in [32] we get that $X$ belongs to the epireflective hull of all discrete spaces. By Proposition 6.3.11 $X$ is $U_{\mathbf{NA}_0}$-injective.

3$\Rightarrow$1: Suppose $X$ is $U_{\mathbf{NA}_0}$-injective in $\mathbf{NA}_0$. In view of Proposition 6.3.8 (2) we can conclude that $X$ is $\zeta$-closed in every Hausdorff non-Archimedean space in which it is embedded. In particular $i : X \to \Pi^\nu_X$ is a $\zeta$-closed embedding. Now if $f$ is any order preserving choice function, Proposition 6.3.14 implies that $f$ converges to some point of $X$.  

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Let $C_{NA_0}$ be the full subconstruct of $NA_0$ consisting of the complete Hausdorff non-Archimedean spaces. $C_{NA_0}$ is the unique $U_{NA_0}$-reflective subconstruct of $NA$. In the next proposition we give an explicit description of the reflection, for a space $X$ in $NA_0$, which is in fact the “unique completion” of $X$.

**Theorem 6.3.16.** Let $X$ be a Hausdorff non-Archimedean space. Let $\hat{X}$ be the set consisting of all order preserving choice functions of $X$. For every $E \in \mathcal{E}_X$, let $P_E$ be the partition given by $E$. We define $\hat{E} = \{(f, g) \in \hat{X} | f(P_E) = g(P_E)\}$. Consider the non-Archimedean space $\hat{X}$ where the structure is given by $D_{\hat{X}} = stack_{\hat{E}} \hat{E}$ and $\hat{E} = \{E | E \in \mathcal{E}_X\}$. We have the following:

1. $\hat{X}$ is a complete Hausdorff non-Archimedean space.
2. $X$ is a $\zeta$-dense subspace of $\hat{X}$.

**Proof.**

1. Suppose that $f, g \in \hat{X}$ are different. Then there is a $P \in \nu_X$ such that $f(P) \neq g(P)$. So $\hat{X}$ is Hausdorff.

Let $\hat{f}$ be an order preserving choice function on $\hat{X}$. We define an order preserving choice function on $X$ as follows. Each $P \in \nu_X$ has a corresponding equivalence relation $E_P$, for which $\hat{E}_P$ has a partition $\hat{P}$ on $\hat{X}$. We define $f : \nu_X \to \nu_X : P \mapsto \hat{f}(P)$. For any $P$ we have that $f \in \hat{f}(P)$. Hence $f$ converges to $f$, so $\hat{X}$ is complete.

2. Consider the following map:

$$j : X \to \hat{X} : x \mapsto (f_x : \nu_X \to \nu_X : P \mapsto [x]_P)$$

Clearly $f_x$ is an order preserving choice function. Since $X$ is Hausdorff $j$ obviously is injective.

Let $E \in \mathcal{E}_X$, clearly $E = (j \times j)^{-1}(\hat{E})$. Therefore $j$ is initial.

Let $f \in \hat{X}$ and let $F_1, F_2 \in \mathcal{E}_X$ which coincide on $j(X)$. There exist $E_1, E_2 \in \mathcal{E}_X$ and the corresponding partitions $P_1, P_2$ such that $\hat{E}_1 \subset F_1$ and $\hat{E}_2 \subset F_2$.

We have that $\hat{E}_1[f] = \{g \in \hat{X} | f(P_1) = g(P_1)\}$. Since $\emptyset \neq f(P_1) \in P_1$ there is an $x \in f(P_1)$ such that $f_x \in \hat{E}_1[f]$, so $(f, f_x) \in \hat{E}_1 \subset F_1$. Analogously there is a $y \in f(P_2)$ for which $(f, f_y) \in \hat{E}_2 \subset F_2$.

Since $F_1, F_2$ coincide on $j(X)$, we have that $(j \times j)^{-1}(F_1) = (j \times j)^{-1}(F_2)$, the latter corresponding to a $P' \in \nu_X$ for which $P_1, P_2 < P'$. Since
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\( f \) is order preserving and because of our choice of \( x \) and \( y \) we have that \( x, y \in f(P') \). Hence \( (f_x, f_y) \in F_2 \) and \( (f, f_x) \in F_2 \). Therefore \( f_x \in F_1[f] \cap F_2[f] \cap j(X) \). So finally \( f \in \zeta_X(j(X)) \).

\[ \square \]

From the previous theorem and starting with \( X \) in \( \text{NA}_0 \), we can conclude that, with respect to the class \( \mathcal{U}_{\text{NA}_0} \) of \( \zeta \)-dense embeddings, \( j : X \rightarrow \hat{X} \) is the unique completion of \( X \). Indeed if \( r_X : X \rightarrow RX \) is the reflection of \( X \) in \( \text{CNA}_0 \), then with the notations of Theorem 6.3.16 we can consider the diagram:

\[
\begin{array}{ccc}
RX & \xrightarrow{r_X} & \hat{X} \\
\downarrow & & \downarrow \\
X & \xrightarrow{j} & \hat{X} \cong RX
\end{array}
\]

Since \( \text{CNA}_0 \) is firm \( \mathcal{U}_{\text{NA}_0} \)-reflective and \( j \) is \( \zeta \)-dense, this means that \( j \) is an isomorphism.

It follows that a uniformly continuous map \( u : X \rightarrow Y \) from a Hausdorff non-Archimedean space \( X \) to a complete Hausdorff non-Archimedean space \( Y \) can be uniquely extended to a uniformly continuous map \( \hat{u} : \hat{X} \rightarrow Y \). We describe this extension explicitly as follows.

Let \( f \in \hat{X} \). For any \( P \in \nu_Y \) we know that \( u^{-1}(P) \in \nu_X \), hence \( f(u^{-1}(P)) \) is a class of \( u^{-1}(P) \) and \( u(f(u^{-1}(P))) \) is a subset of a class from \( P \), which we will write as \( f_u(P) \). This defines an order preserving choice function as follows:

\[ f_u : \nu_Y \rightarrow \cup \nu_Y : P \mapsto f_u(P) \]

The extension of \( u \) is then given by:

\[ \hat{u} : \hat{X} \rightarrow Y : f \mapsto \lim f_u \]

where \( \lim f_u \) is the unique limit point of \( f_u \), which exists since \( Y \) is Hausdorff and complete.

**Example 6.3.17.** [The space non-Archimedean space from Example 6.3.2 is complete.]

We consider the non-Archimedean space \( X \) as defined in Example 6.3.2, i.e. \( X = \{1, 2, 3, 4, 5\} \) and \( \mu_X = \text{stack}_\nu \{P_1, P_2\} \), where

\[
\begin{align*}
P_1 &= \{\{1, 2, 3\}, \{4\}, \{5\}\} \\
P_2 &= \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}
\end{align*}
\]
6. Non-Archimedean uniform pre-nearness spaces

We will show that this space is a complete space and give a description of all order-preserving choice functions. First we note that \( P = \{\{1,2,3\},\{4,5\}\} \) is a partition, refined by both \( P_1 \) and \( P_2 \). Hence if \( f \) is an order-preserving choice function then \( f(P_1) = \{1,2,3\} \) implies that \( f(P_2) \) is one of \( \{1\}, \{2\}, \{3\} \) and if \( f(P_1) \) is one of \( \{4\}, \{5\} \) then \( f(P_2) \) must be \( \{4,5\} \). These possibilities generate all order-preserving choice functions. Since these choices imply the convergence of every order-preserving choice function, we have that the space \( X \) is complete.

\[ \triangleright \]

**Example 6.3.18.** [A finite Hausdorff non-Archimedean space, which is not complete.]

Let \( X \) be the set \( \{1,2,3\} \). We define a Hausdorff non-Archimedean space \( \hat{X} \) on \( X \) by means of the collection \( \nu_X = \{\{\{1\}\}, \{\{2,3\}\}, \{\{3\}\}, \{\{1,2\}\}, \{X\}\} \). For this space the order-preserving choice function \( f \) for which

\[
\begin{align*}
  f(\{\{1\}\}, \{\{2,3\}\}) &= \{1\} \\
  f(\{\{3\}\}, \{\{1,2\}\}) &= \{3\} \\
  f(\{X\}) &= X
\end{align*}
\]

does not converge, hence \( \hat{X} \) is not complete. In order to construct the completion of this space we note that \( \Pi \nu_X \) is isomorphic to \( D_2 \times D_2 \). Hence we have the embedding \( i : X \rightarrow D_2 \times D_2 \), which is not \( \zeta \)-closed. Since \( X \) has three points, it follows that \( i \) is \( \zeta \)-dense. Thus the completion \( \hat{X} \) is isomorphic to \( D_2 \times D_2 \).

\[ \triangleright \]

### 6.3.2 The case of non-Archimedean uniform spaces

In this last section we will show that the previously described completion is in fact a generalization of the classical completion of a Hausdorff non-Archimedean uniform space, as described in [13],[49],[50].

Let \( X \) be a Hausdorff non-Archimedean uniform space as introduced by A.F. Monna in [36]. We know that it is a Hausdorff non-Archimedean space in our sense.

**Proposition 6.3.19.** Let \( X \) be a non-Archimedean uniform space and let \( F \) be a minimal small filter in \( X \). Then there is a unique element \( (z_P)_{P \in \nu_X} \in \Pi \nu_X \) such that \( F = \text{stack} \{z_P | P \in \nu_X\} \).
Proof. We only have to check uniqueness since such a collection exists because \( \mathcal{F} \) is minimal. If \( \mathcal{F} = \text{stack} \{ z_P | P \in \nu_X \} = \text{stack} \{ z'_P | P \in \nu_X \} \), then for every \( P \) \( z_P \cap z'_P \) is nonempty since \( \mathcal{F} \) is a filter. Hence \( z_P = z'_P \).

Proposition 6.3.20. Let \( X \) be a non-Archimedean uniform space. There is a one to one correspondence between the order preserving choice functions of \( X \) and the minimal small filters of \( X \).

Proof. Let \( A \) be the set of all order preserving choice functions of \( X \) and let \( B \) denote the set of all its minimal small filters. The following maps describe the needed one to one correspondence.

\[
F : A \to B : f \mapsto \mathcal{F}_f
\]

where \( \mathcal{F}_f = \text{stack} \{ z_P | P \in \nu_X \} \) with \( z_P = f(P) \) for \( P \in \nu_X \).

\[
G : B \to A : \mathcal{F} \mapsto f_{\mathcal{F}}
\]

where \( f_{\mathcal{F}}(P) = z_P \) is uniquely defined by Proposition 6.3.19.

By definition \( \mathcal{F}_f \) contains arbitrarily small sets. For \( P, Q \in \nu_X \) we have \( z_{P \wedge Q} \subseteq z_P \cap z_Q \), so \( \mathcal{F}_f \) is a filter. By the same argument as in the proof of Proposition 6.3.19 one has that \( \mathcal{F}_f \) is minimal.

\( f_{\mathcal{F}} \) is a well-defined choice function by Proposition 6.3.19. For \( P, Q \in \nu_X \) we have \( z_{P \wedge Q} = z_P \cap z_Q \). It follows that \( f_{\mathcal{F}} \) is order preserving.

After a simple verification one sees that \( F \) and \( G \) are bijective and inverse to one another.

Since, through the bijections \( F \) and \( G \), convergent order preserving choice functions correspond to convergent minimal small filters we can conclude the following.

Corollary 6.3.21. The completion as developed in Theorem 6.3.16, when applied to a Hausdorff non-Archimedean uniform space, reduces to the classical completion.

In fact this is not really a surprising result, it also follows from the next two observations.
6. Non-Archimedean uniform pre-nearness spaces

The category \( \text{NAUnif} \) of non-Archimedean uniform spaces is a hereditary coreflective full subcategory of the well-fibred topological construct \( \text{NA} \), moreover the discrete non-Archimedean spaces form a class of \( \mathcal{U}_{\text{NA} 0} \)-injective \( \text{NAUnif} \)-objects which \( \text{Emb} \ \text{NA} 0 \)-cogenerates \( \text{NA} 0 \). Since the latter is co-well-powered and its regular closure \( \zeta \) is weakly hereditary, one can apply Theorem 4.2.17. Hence both \( \text{NA} 0 \) and \( \text{NAUnif} 0 \) have a subcategory of complete objects \( \text{CNA} 0 \) resp. \( \text{CNAUnif} 0 \). It also follows that \( \text{CNA} 0 \cap \text{NAUnif} 0 \subset \text{CNAUnif} 0 \) and that the complete objects are exactly the \( \zeta \)-closed subspaces of products (taken in \( \text{NA} 0 \) resp. \( \text{NAUnif} 0 \)) of discrete spaces.

On the other hand if \( X \) is a \( \text{NAUnif} 0 \)-object and one constructs its \( \text{NA} 0 \)-completion \( \hat{X} \) as described in Theorem 6.3.16 one easily sees that \( \hat{X} \) is also an \( \text{CNAUnif} 0 \)-object. Therefore In this case one has \( \text{CNA} 0 \cap \text{NAUnif} 0 = \text{CNAUnif} 0 \).

6.3.3 Completeness versus s-compactness

Now that we have a well-behaved completion theory for non-Archimedean spaces, we investigate how these complete non-Archimedean spaces relate to s-compactness as we introduced in Definition 3.3.4.

**Proposition 6.3.22.** Let \( X \) be a Hausdorff non-Archimedean space, such that the induced closure space \( CX \) is s-compact. Then for \( P \in \nu X \) we have \( |P| \leq 2 \).

**Proof.** Suppose \( P \in \nu X \) with \( |P| > 2 \) and consider

\[
A = \{ A \in \mathcal{C}(C X) | \exists P \in P : X \setminus P \subset A \}
\]

Clearly \( A \) is a stack of \( \mathcal{C}(X) \). If there would be a clopen set \( A \) of \( CX \) such that \( A, X \setminus A \in A \), then there would be \( P_1, P_2 \in P \) for which \( X \setminus P_1 \subset A \) and \( X \setminus P_2 \subset X \setminus A \). Hence \( P_1 \cup P_2 = X \) and since \( P \) is a partition of \( X \), \( |P| \leq 2 \), which contradicts our supposition. Therefore \( A \in \chi_{\mathcal{C}(C X)} \), so there exists a prime stack \( A^* \) which contains \( A \) and which converges. Hence the following contradiction

\[
\emptyset \neq \cap A^* \subset \cap A = \cap_{P \in P} X \setminus P = \emptyset
\]

Thus we conclude that \( |P| \leq 2 \). 

This leads us to the following definition.
6. Non-Archimedean uniform pre-nearness spaces

**Definition 6.3.23.** A non-Archimedean space $X$ such that $P \in \nu_X$ implies $|P| \leq 2$ will be called **2-bounded.**

**Proposition 6.3.24.** Let $X$ be a complete, 2-bounded Hausdorff non-Archimedean space. Then $CX$ is s-compact.

*Proof.* Let $\mathcal{A}$ be a prime stack of $CX$. We define a choice function $f$ which maps each $P \in \nu_X$ to the unique element of $\mathcal{A} \cap P$. Since $X$ is 2-bounded this choice function is order-preserving. By completeness of $X$ there is limit point $x \in X$ of $f$. If $\mathcal{A}$ does not convergent to $x$, then there is a clopen set $A$ such that $x \in A, X \setminus A \in \mathcal{A}$. Since $CX$ is induced by the non-Archimedean space $X$ there is a $P \in \nu_X$ for which $[x]_P \subset A$. Since $f$ converges to $x$ we have that $f(P) = [x]_P \subset A$, therefore $A$ and $X \setminus A$ are in $\mathcal{A}$, which is impossible since $\mathcal{A}$ is a prime stack. Thus we conclude that $\mathcal{A}$ converges to $x$. 

One might expect that for a Hausdorff non-Archimedean space $X$ s-compactness of $CX$ is equivalent with completeness and 2-boundedness of $X$. However, the next example shows that this is not true.

**Example 6.3.25.** [A 2-bounded non-Archimedean space which is not complete, but which does induce an s-compact closure space.]

Consider the space $X$ introduced in Example 6.3.18: we endow $X = \{1, 2, 3\}$ with the Hausdorff non-Archimedean structure generated by $\nu_X = \{\{1\}, \{2, 3\}, \{3\}, \{1, 2\}, \{X\}\}$. From Example 6.3.18 we now that this is not a complete space. Obviously, it is 2-bounded. The underlying closure space $CX$ has the following clopen sets $CO(CX) = \{\{1\}, \{2, 3\}, \{3\}, \{1, 2\}, \emptyset, X\}$ and is in fact isomorphic to the closure space from Example 1.2.22. The only prime stacks are

$$\\{\{1\}, \{1, 2\}, X\}$$

$$\\{\{3\}, \{2, 3\}, X\}$$

$$\\{\{1, 2\}, \{2, 3\}, X\}$$

which are all convergent, hence $CX$ is s-compact.
A. Topology with Maple

Appendix A

Topology with Maple

This appendix should contain a listing of Maple 6 procedures, which were used to produce and examine many examples concerning some of the topics of this thesis, however, since there are about thirty pages of listing, I refer to my homepage http://homepages.vub.ac.be/~didesen/Math.htm where the file topotheoryv3.mpl is downloadable.

The main limitation of this implementation of topology in maple is the fact that only (small) finite spaces can be considered, however it comes in handy when searching for examples. This programming should not be considered as a part of this thesis, however since I have put quite some free time in it and I did use it quit often in my research, I feel it should be included as an appendix.
Appendix B

Nederlandse samenvatting

Binnen de wiskunde en andere wetenschappen komen vaak modellen voor waarbij men een tralietheoretische aanpak hanteert. Zo worden bijvoorbeeld tralies gebruikt in de formele analyse van concepten [29] of in de beschrijving van fysische systemen [16, 5]. In vele van deze theorieën wordt aan een tralie op een natuurlijke wijze een sluiting operator geassocieerd. Deze operator is echter niet topologisch van aard. Hiermee bedoelen we dat de operator wel idempotent, extensief en monotoon is en dat ø gesloten is, maar dat de operator niet additief is, hetgeen vertaald wordt door het feit dat een eindige unie van gesloten delen niet gesloten hoeft te zijn. In de wiskunde kent men nog andere gevallen van zulke sluitingen die niet afkomstig zijn van een tralie, zo is er bijvoorbeeld de convexe sluiting.

Omdat deze sluitingen van groot belang zijn, hebben ze geleid tot de categorie **Cl** van closure ruimten en continue functies. In het eerste hoofdstuk van deze doctoraatsverhandeling wordt deze categorie ingeleid. Net zoals topologische ruimten kunnen closure ruimten worden beschreven aan de hand van gesloten delen, open delen, een sluiting of een familie omgevingenstacks. Bovendien is **Cl** een well-fibred topological construct. Uit het werk van T. Marny [34] weten we dat er een volle deelcategorie **Cl**ₐ van **T**₀ objecten van **Cl** bestaat. Naar analogie met topologische ruimten, voeren we ook de hogere separatie-eigenschappen **T**₁ en **T**₂ in voor closure ruimten. Daarna worden er op dezelfde manier als in **Top**, aan de hand van clopen delen, nuldimensionale closure ruimten gedefinieerd. Deze geven aanleiding tot de volle bifreflectieve deel categorie **0Cl** van **Cl**. Nadien worden, tevens met behulp van clopen delen, de begrippen samenhang en totale onsamenhang voor closure ruimten geïntroduceerd. We tonen aan dat de
deelcategorieën van samenhangende closures en van totaal onzamenhangende closures een connectedness, respectievelijk een disconnectedness vormen in de zin van [10]. Bovendien is de volle deelcategorie TDiscCl van totaal onzamenhangende closure ruimten een epireflectieve deelcategorie. Net zoals in het topologisch geval wordt de epireflectie van een closure ruimte beschreven als een quotiënt, t.o.v. een samenhangs-equivalentierelatie $K$. Hoewel hierdoor het parallelisme met de categorie van de topologische ruimten zeer groot is toont het voorbeeld van een eindige totaal onzamenhangende Hausdorff closure ruimte die niet nuldimensionaal is, aan, dat er toch sterke verschillen bestaan met het topologisch geval.

Eenmaal de elementaire begrippen i.v.m. closure ruimten besproken zijn wordt in Hoofdstuk 1 een aantal voorbeelden uitgewerkt. Het eerste voorbeeld bestaat uit de sluitingen vect, aff en conv uit de lineaire algebra en de convex analyse. De verschillende separatie-eigenschappen van deze sluitingen worden bestudeerd en er wordt aangetoond dat conv een nuldimensionale sluiting is. Een ander voorbeeld wordt gegeven door de sluiting die werd ingevoerd in [29] om in het kader van formele contexten de begrippen “concept” en “concepten tralie” te introduceren. Een laatste voorbeeld dat in dit hoofdstuk wordt behandeld is de theorie van de state property systemen. In deze theorie worden fysische systemen beschreven aan de hand van toestanden van het systeem en een tralie van eigenschappen [5]. Een state property systeem is een tripel bestaande uit een verzameling toestanden $\Sigma$, een tralie van eigenschappen $\mathcal{L}$ en een afbeelding $\xi : \Sigma \rightarrow \mathcal{P}(\mathcal{L})$ waarbij, voor iedere toestand $p \in \Sigma$, $\xi(p)$ de verzameling van alle actuele eigenschappen voorstelt, i.e. de eigenschappen die waar zijn als het systeem zich in toestand $p$ bevindt. In [6] werd aangetoond dat de categorie van state property systemen equivalent is met de categorie Cl. Met ieder state property systeem wordt een closure ruimte geadjudeerd (de eigenclosure). Bovendien bleek dat een aantal fysisch geïnspireerde eigenschappen overeenkomen met gekende topologische begrippen. Zo komen de zogenoemde atomististische state property systemen overeen met de $T_1$ closure ruimten [47].

In Hoofdstuk 2 wordt de equivalentie tussen de state property systemen en de closure ruimten benut om de klassieke eigenschappen van een state property systeem te bestuderen. Een van de belangrijkste verschillen tussen quantum-mechanische systemen en klassieke systemen is het bestaan van superposities. In het kader van state property systemen wordt het begrip van “super selection rule” gebruikt om een notie van z-klassiek state property systeem in te voeren, namelijk een state property systeem waarvoor iedere twee eigenschappen gescheiden worden door een superselectieregel, i.e. voor elke twee eigenschap-
pen mag er geen superpositie bestaan. Er wordt aangetoond dat de eigenclosure die overeenkomt met een s-klassiek state property systeem topologisch is. De categorie van de s-klassieke state property systemen is dus equivalent met die van de topologische ruimten.

Men kan ook een andere vorm van klassieke state property systemen beschouwen, namelijk door middel van de zogenaamde deterministisch klassieke eigenschappen (d-klassiek). Een eigenschap \( a \) heet d-klassiek indien er een andere, complementaire, eigenschap \( a^c \) bestaat zodat altijd juist een van beide actueel en zodanig dat ze gescheiden worden door een superpositieregel. Uit deze definitie volgt dat de d-klassieke eigenschappen van een state property systeem overeenkomen met de clopen delen van de eigenclosure. Een state property systeem heet zuiver niet-klassiek indien er geen d-klassieke eigenschappen bestaan buiten de eigenschappen \( \top \) (die altijd actueel is) en \( \bot \) (die nooit actueel is). Aldus bekomt men een equivalentie tussen de categorie van zuiver niet-klassieke state property systemen en die van de samenhangende closure ruimten. Anderzijds heet een state property systeem totaal klassiek indien de tralie van eigenschappen geen enkel niet-triviaal segment bevat dat zuiver niet-klassiek is. De categorie van de atomistische totaal klassieke state property systemen is equivalent met de categorie \( \text{TDiscCl} \).

Om dit hoofdstuk af te sluiten wordt gebruik gemaakt van een aantal topologische methoden om uit een state property systeem informatie te halen omtrent haar klassieke en niet-klassieke eigenschappen. Zo biedt de \( \text{TDiscCl} \)-epireflectie van de eigenclosure van een state property systeem een mogelijkheid om dit systeem op te splitsen in een aantal zuiver niet-klassieke subsystemen (de samenhangskomponenten) en een overkoepelend totaal klassiek systeem (de \( \text{TDiscCl} \)-epireflectie). De \( \text{0Cl} \)-bireflectie laat op haar beurt toe om uit een state property systeem juist die eigenschappen af te zonderen die superposities zijn van d-klassieke eigenschappen. Al deze resultaten werden bekomen in samenwerking met D. Aerts en A. De Groot-Van der Voorde en zijn verschenen in [7] en [8].

In het derde hoofdstuk van dit werk worden verschillende representaties van closure ruimten, aan de hand van tralies en partiële ordes, bestudeerd. Eerst wordt naar analogie met de theorie van frames en sobere topologische ruimten (zie bijvoorbeeld [14]) een dualiteit tussen de categorie van complete tralies met \( \vee \)-bewarende afbeeldingen en de volle deelcategorie \( \text{CCl}_0 \) van \( \text{Cl}_0 \) aangetoond. Deze deelcategorie wordt gegeven door zekere complete \( T_0 \) closure ruimten, waarin bepaalde stacks met open basis moeten convergeren. De complete \( T_0 \) closures worden tevens gekarakteriseerd door het feit dat ieder niet-leeg, gesloten deel de sluiting is van een uniek punt, aldus worden ze ook punt-closure.
ruimten genoemd. De functoren van de hierboven vermelde dualiteit zijn formeel dezelfde als de functoren die de dualiteit beschrijven tussen sobere topologische ruimten en spatial frames. Hoewel er dus een formele en categorische gelijkenis is tussen beide dualiteiten, kan men de punt-closure ruimten niet als veralgeme- mening van sobere topologische ruimten beschouwen. Immers, iedere Hausdorff topologische ruimte (met meer dan twee punten) is sober maar kan nooit een punt-closure ruimte zijn. Deze resultaten werden samen met E. Giuli en E. Lowen-Colebunders opgesteld en zijn verschenen in [20].

Een andere reeks representaties van closure ruimten wordt gegeven aan de hand van de theorie ontwikkeld door M. Erné [26]. Hier wordt middels een invariante selectie $\Sigma$ op een tralie een equivalentie beschreven tussen een categorie $L_\Sigma$ van complete tralies $L$ waarvoor $\Sigma(L)$ een $\lor$-basis is en een volle deelcategorie $C_\Sigma$ van $Cl_0$. In dit werk breiden we deze equivalenties uit op volgende wijze. De invariante selectie induceert een natuurlijke onderliggende functor $U_\Sigma : L_\Sigma \to Set$, zodat $L_\Sigma$ een construct wordt. Gebruik makend van de notie van concrete equivalentie, ingevoerd door H.E. Porst in [40], wordt getoond dat de equivalentie beschreven in [26], een concrete equivalentie is wanneer men $L_\Sigma$ beschouwd als construct. Er worden hierna drie voorbeelden behandeld. Indien $\Sigma(L) = A_L$ (de verzameling atomen van $L$) bekomen men de concrete equivalentie tussen de categorie van atomistische tralies en die van $T_1$ closure ruimten, die aan de basis ligt van de equivalentie tussen atomistische state property systemen en $T_1$ closure ruimten [47]. Als men voor $\Sigma(L)$ de $\lor$-irreducibele elementen van $L$ kiest bekomen men een equivalentie van $L_\Sigma$ met sobere closure ruimten, die wel als veralgemening van sobere topologische ruimten kunnen gezien worden. Indien men $\Sigma(L) = L \setminus \{ \bot \}$ stelt, dan bekomen men een equivalentie tussen $L_\Sigma$ en de categorie van punt-closure ruimten $CCl_0$.

Het laatste dat in Hoofdstuk 3 wordt voorgesteld, is een vorm van de Stone representatiestelling in de context van closure ruimten. Het blijkt dat in het geval een closure ruimte $X$ de verzameling $CO(X)$ van clopen delen geen Booleaanse tralie is maar slechts een partiële orde, voorzien van een complementatie. In [35] wordt een dualiteit aangetoond tussen een deelcategorie van closure ruimten en de categorie van partiële ordes uitgerust met een complementatie. We karakterizeren deze klasse van closure ruimten als nulldimensionale Hausdorff closures, die compact zijn in die zin dat bepaalde priem stacks moeten convergeren (s-compact). Er blijkt dat deze vorm van compactheid geen veralgemening is van compactheid voor topologische ruimten, zo bestaan er bijvoorbeeld eindige topologische ruimten die niet s-compact zijn. De deelcategorie van $0Cl_0$ bestaande uit compacte objecten is wel epireflectief en de epireflectie is
bovendien een inbedding zodat men wel een compactificatie bekomt.

In het vierde hoofdstuk wordt compleetheid nader bekeken. De zoektocht naar
een categorische formulering van het begrip compleetheid heeft G.C.L. Brümmer
en E. Giuli [18, 19] geleid tot de notie van firm $U$-reflectieve deelcategorie van een
categorie $X$, die als deelcategorie van complete objecten kan worden beschouwd.
Zulk een firm $U$-reflectieve deelcategorie een reflectieve deelcategorie waar, voor
echter object $X$, de reflectie $r_X : X \rightarrow RX$ in $U$ zit en zodat de completie uniek
is in die zin dat een morfisme $u : X \rightarrow Y$ een $U$-morfisme is enkel en alleen
indien $Ru : RX \rightarrow RY$ een isomorfisme is. In het geval van een complete
well-powered construct $X$ en indien $U$ de klasse van epimorfe inbeddingen is
dan bestaat er hoogstens een firm $U_X$-reflectieve deelcategorie van $X$. Indien ze
bestaat, is ze gelijk aan de volle deelcategorie van $U_X$-injectieve objecten en als
$X$ wordt gecogenerereerd door een klasse $P$ van $U_X$-injectieve objecten, dan is de
firm $U_X$-reflectieve deelcategorie de epireflectieve omhullende $E_X(P)$.

Deze resultaten worden benut om aan te tonen dat de categorie $\text{Cl}_b$, die geco-
genereerd is door de Sierpinski ruimte $S_2$, een unieke firm $U_{\text{Cl}_b}$-reflectieve deel-
categorie bezit. Door gebruik te maken van de reguliere sluiting geassocieerd
met $\text{Cl}_b$ (de $b$-closure), blijkt dat de objecten in deze deelcategorie de regulier
gesloten deelruimten zijn van machten van $S_2$, bovendien bekomen men aldus
de categorie $\text{CCl}_b$ van punt-closures. Hieruit blijkt dus dat deze ruimten wel
degelijk de complete $T_0$ closure ruimten zijn. We tonen ook aan dat, net zoals in
het topologisch geval, de categorie $0\text{Cl}_b$ van $T_0$ multidimensionale closure ruimten
geen $U_{0\text{Cl}_b}$-reflectieve deelcategorie bezit. Dit zal ons ertoe leiden om in de vol-
gende hoofdstukken de niet-Archimedische ruimten in te voeren.

Hierna bekijken we de problematiek van compleetheid in het kader van een core-
reflectieve, hereditaire deelcategorie $C$ van een well-fibred topological construct
$X$. Indien de categorie $T_0 C$ van $T_0$ objecten van $X$ gecogenerereerd wordt door
en klasse $P$ van $C$-objecten die $UT_0$-injectief zijn, dan bezitten zowel $T_0 X$
als $T_0 C$ een deelcategorie van complete objecten, die gegeven worden door de
epireflectieve omhullende $E_{T_0 X}(P)$, resp. $E_{T_0 C}(P)$. Om deze stelling te illus-
treren beschouwen we een familie van reflectieve, hereditaire deelcategorieën
$\text{Tight}(\alpha)$ van $\text{Cl}$. In dit geval worden de complete objecten van $\text{Tight}_b(\alpha)$
gegeven door de $b$-gesloten deelruimten van machten van $S_2$, waarbij de machten
deze keer genomen worden in $\text{Tight}_b(\alpha)$. Een ander voorbeeld wordt gegeven
door de categorie $\text{Top}$ die een reflectieve deelcategorie is van $\text{Cl}$. In dit geval
bezitten zowel $\text{Top}_b$ als $\text{Cl}_b$ een deelcategorie van complete objecten, namelijk
$\text{Sob}$ (de sobere topologische ruimten) resp. $\text{CCl}_b$. Bovendien zijn deze com-
plete objecten in beide gevallen $b$-gesloten deelruimten van machten van $S_2$. 
het verschil is echter te wijten aan het feit dat producten in Top en Cl anders worden gevormd.

Uit het vorige weten we dat de categorie 0Cl₀ van T₀ nuldimensionale closure ruimten geen L₀Cl₀-reflectieve deelcategorie bezit. Omdat deze ruimten toch belangrijk zijn, bijvoorbeeld in het kader van de Stone representatiestelling voor closure ruimten, wensen we toch over een zeker compleetheidsbegrip te kunnen spreken. Hiervoor zullen we een uniform kader trachten te vinden waarin deze ruimten voorstelbaar zijn en waarin men toch een completietheorie kan beschrijven.

Ons baserend op het werk van Herrlich [31] omtrent (pre-)nearness ruimten voeren we in Hoofdstuk 5 de categorie UPNear van uniforme pre-nearness ruimten in. Voor deze ruimten geven we equivalente beschrijvingen aan de hand van uniforme overdekkingen, small collecties, near collecties, entourages en families pseudometrieken. Er wordt aangetoond dat Unif een coreflectieve deelcategorie is van UPNear. Bovendien betaat er een canonische functor van UPNear naar Cl, zodat men over uniformiseerbare closure ruimten kan spreken. Er blijkt dat deze closure ruimten juist de volledig reguliere closure ruimten zijn. Een speciale deelklasse van volledig reguliere closures wordt gevormd door de nuldimensionale closures, dit zal in het laatste hoofdstuk tot de invoering leiden van niet-Archimedische ruimten.

In [53] wordt een representatie gegeven van pre-nearness ruimten door middel van een adjunctie met de categorie PGrL van pre-grill lattices. Om Hoofdstuk 5 af te sluiten wordt getoond dat deze adjunctie kan herleid worden tot een adjunctie tussen UPNear en de categorie UPGrL van uniforme pre-grill lattices.

In het laatste hoofdstuk van deze thesis wordt de categorie NA van niet-Archimedische ruimten ingevoerd en bestudeerd. Dit zijn uniforme pre-nearness ruimten waarvan de structuur volledig bepaald wordt door middel van partities, equivalentierelaties of ultra-pseudometrieken. De canonische closure ruimte geïnduceerd door een niet-Archimedische ruimte is nuldimensionaal, en omgekeerd is iedere nuldimensionale closure niet-Archimedisch uniformiseerbaar.

Er wordt tevens getoond dat de eerder vermelde representatie van UPNear aan de hand van UPGrL, in het niet-Archimedisch geval kan herleid worden tot een adjunctie tussen NA en de categorie van niet-Archimedische pre-grill lattices.
Hierna wordt gezocht naar een compleetie van een niet-Archimedische ruimte. Er blijkt dat $\mathbf{NA}_0$ (de categorie van $T_0$-objecten van $\mathbf{NA}$) een unieke firm $\mathcal{U}_{\mathbf{NA}_0}$-reflectieve deelcategorie van complete objecten bezit. Omdat de refinement relatie $<$ op collectie $\nu_X$ van uniforme partities van een niet-Archimedische ruimte $X$, een partiële orde is kan deze notie van compleetheid gekarakteriseerd worden aan de hand van ordebewarende keuzefuncties. Een ordebewarende keuzefunctie is een functie $f : \nu_X \to \cup \nu_X$ die met elke partitie $P \in \nu_X$ een van haar klassen $f(P)$ associeert, zodanig dat als $P < Q$ geldt dan ook $f(P) \subset f(Q)$. Een niet-Archimedische ruimte $X$ is dan compleet als en slechts als iedere ordebewarende keuzefunctie $f$ convergeert, i.e. $\cap_{P \in \nu_X} f(P) \neq \emptyset$. Uit het feit dat de categorie van niet-Archimedische uniforme ruimten een coreflectieve deelcategorie is van $\mathbf{NA}$, kan men tonen dat in het geval van zulk een uniforme Hausdorff ruimte de compleetie in $\mathbf{NA}_0$ samenvalt met haar klassieke compleetie. Deze resultaten, die verkregen werden in samenwerking met E. Colebunders werden gepubliceerd in [21]. Uiteindelijk wordt tevens getoond dat compleetheid en (totaal) begrensdeheid van een niet-Archimedische ruimte niet equivalent is met s-compactheid van de onderliggende nuldimensionale closure ruimte.
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