On the Normalizers of Subgroups in Integral Group Rings

Andreas Bächle

Vrije Universiteit Brussel

DMV-PTM Mathematical Meeting

17. - 20.09.2014, Poznań
Notations

\( G \) \hspace{1cm} \text{finite group}
Notations

$G$ finite group

$R$ commutative ring with identity element 1
Notations

\( G \) finite group
\( R \) commutative ring with identity element \( 1 \)
\( RG \) group ring of \( G \) with coefficients in \( R \)
Notations

$G$ finite group

$R$ commutative ring with identity element 1

$RG$ group ring of $G$ with coefficients in $R$

$U(RG)$ group of units of $RG$
Notations

$G$  finite group
$R$  commutative ring with identity element 1
$RG$ group ring of $G$ with coefficients in $R$
$U(RG)$ group of units of $RG$
The normalizer problem

\[ N_{U(RG)}(G) \]
The normalizer problem

\[ N_{U(RG)}(G) \supseteq \]
The normalizer problem

$N_{U(RG)}(G) \supseteq G$
The normalizer problem

\[ N_{U(RG)}(G) \supseteq G \cdot Z(U(RG)) \]
The normalizer problem

$$N_{U(RG)}(G) \supseteq G \cdot Z(U(RG))$$

We say that $G$ (together with the ring $R$) has the normalizer property, if
The normalizer problem

\[ N_{U(RG)}(G) \supseteq G \cdot Z(U(RG)) \]

We say that \( G \) (together with the ring \( R \)) has the normalizer property, if

\[ N_{U(RG)}(G) = G \cdot Z(U(RG)), \]
The normalizer problem

\[ N_{U(RG)}(G) \supseteq G \cdot Z(U(RG)) \]

We say that \( G \) (together with the ring \( R \)) has the normalizer property, if

\[ N_{U(RG)}(G) = G \cdot Z(U(RG)) \]

abbreviated (NP).
is an automorphism problem

\[ u \in \text{N}_{U(RG)}(G) \]
is an automorphism problem

\[ u \in N_U(RG)(G) \leadsto \]
is an automorphism problem

\[ u \in N_{U(RG)}(G) \mapsto \text{conj}(u) : G \to G, \quad g \mapsto g^u \]
is an automorphism problem

\[ u \in N_{U(RG)}(G) \quad \mapsto \quad \text{conj}(u) : \quad G \rightarrow G \]

\[ g \mapsto g^u \]

Set \( \text{Aut}_{RG}(G) = \{ \text{conj}(u) \mid u \in N_{U(RG)}(G) \} \leq \text{Aut}(G) \)
is an automorphism problem

\[ u \in N_{U(RG)}(G) \leadsto \text{conj}(u) : G \rightarrow G \]
\[ g \mapsto g^u \]

Set \( \text{Aut}_{RG}(G) = \{ \text{conj}(u) \mid u \in N_{U(RG)}(G) \} \leq \text{Aut}(G) \)

**Lemma**

\((NP)\) holds for \(G\) \iff \(\text{Aut}_{RG}(G) = \text{Inn}(G)\).
is an automorphism problem

\[ u \in N_{U(RG)}(G) \mapsto \text{conj}(u): \quad G \rightarrow G \]
\[ g \mapsto g^u \]

Set \( \text{Aut}_{RG}(G) = \{ \text{conj}(u) \mid u \in N_{U(RG)}(G) \} \leq \text{Aut}(G) \)

Lemma

\((NP)\) holds for \( G \iff \text{Aut}_{RG}(G) = \text{Inn}(G).\)

Hence we want to control
\[ \text{Inn}(G) \leq \text{Aut}_{RG}(G) \leq \text{Aut}(G). \]
We want to control

\[ \text{Inn}(G) \leq \text{Aut}_{RG}(G) \leq \text{Aut}(G). \]
Some tools

We want to control

\[ \text{Inn}(G) \leq \text{Aut}_{RG}(G) \leq \text{Aut}(G). \]

**Lemma**

\[ \text{Aut}_{\mathbb{Z}G}(G)/\text{Inn}(G) \text{ is an elementary-abelian 2-group.} \]  

(Krempa)
Some tools

We want to control

\[ \text{Inn}(G) \leq \text{Aut}_{RG}(G) \leq A \leq \text{Aut}(G). \]

Lemma

\[ \text{Aut}_{\mathbb{Z}G}(G)/\text{Inn}(G) \text{ is an elementary-abelian } 2\text{-group.} \]  
\text{(Krempa)}
Some tools

We want to control

$$\text{Inn}(G) \leq \text{Aut}_{RG}(G) \leq A \leq \text{Aut}(G).$$

Candidates for $A$:

**Lemma**

$$\text{Aut}_{\mathbb{Z}G}(G)/\text{Inn}(G) \text{ is an elementary-abelian 2-group.} \quad \text{(Krempa)}$$
We want to control

\[ \text{Inn}(G) \leq \text{Aut}_{RG}(G) \leq A \leq \text{Aut}(G). \]

Candidates for \( A \):

\[ \text{Aut}_c(G) = \{ \varphi \in \text{Aut}(G) \mid \forall x \in G: \varphi(x) \sim x \} \]

**Lemma**

\[ \text{Aut}_{\mathbb{Z}G}(G)/\text{Inn}(G) \text{ is an elementary-abelian 2-group.} \]  
(Krempa)
Some tools

We want to control

\[
\text{Inn}(G) \leq \text{Aut}_{RG}(G) \leq A \leq \text{Aut}(G).
\]

Candidates for \( A \):

\[
\text{Aut}_c(G) = \{ \varphi \in \text{Aut}(G) \mid \forall x \in G: \varphi(x) \sim x \}
\]

and – if \( G \) is finite and \( R \) is \( G \)-adapted –

\[
\text{Aut}_{Col}(G) = \left\{ \varphi \in \text{Aut}(G) \mid \forall p \forall P \in \text{Syl}_p(G) \exists g \in G: \varphi|_P = \text{conj}(g)|_P \right\}
\]

Lemma

\[
\text{Aut}_{ZG}(G)/\text{Inn}(G) \text{ is an elementary-abelian 2-group.} \quad (\text{Krempa})
\]
Some tools

We want to control

\[ \text{Inn}(G) \leq \text{Aut}_{RG}(G) \leq A \leq \text{Aut}(G). \]

Candidates for \( A \):

\[ \text{Aut}_c(G) = \{ \varphi \in \text{Aut}(G) \mid \forall x \in G : \varphi(x) \sim_G x \} \]

and – if \( G \) is finite and \( R \) is \( G \)-adapted –

\[ \text{Aut}_{Col}(G) = \left\{ \varphi \in \text{Aut}(G) \mid \forall p \forall P \in \text{Syl}_p(G) \exists g \in G : \varphi|_P = \text{conj}(g)|_P \right\} \]

Lemma

\( \text{Aut}_Z(G)/\text{Inn}(G) \) is an elementary-abelian 2-group. (Krempa)

\( \text{Aut}_{RG}(G) \leq \text{Aut}_c(G), \quad \text{Aut}_{RG}(G) \leq \text{Aut}_{Col}(G). \)
Positive results

Theorem

\((NP)\) holds for

- finite groups with normal Sylow \(2\)-subgroups (Jackowski, Marciniak, 1987)
- finite groups \(G\) with \(R(G) \neq 1\) (Li, Parmenter, Sehgal, 1999)
- finite quasi-nilpotent groups, finite \(2\)-constrained groups \(G\), where \(G/\text{O}_2(G)\) has no chief factor of order 2 (Hertweck, Kimmerle, 2001)
- locally nilpotent groups, periodic groups with normal Sylow \(2\)-subgroup (Jespers, Juriaans, de Miranda, Rogerio, 2002)
Positive results

Theorem

(NP) holds for

✓ finite groups with normal Sylow 2-subgroups
  (Jackowski, Marciniak, 1987)
Positive results

Theorem

\((NP)\) holds for

✓ finite groups with normal Sylow 2-subgroups
  
  (Jackowski, Marciniak, 1987)

✓ finite groups \(G\) with \(R(G) \neq 1\)
  
  (Li, Parmenter, Sehgal, 1999)
Positive results

Theorem

(NP) holds for

✓ finite groups with normal Sylow 2-subgroups
  (Jackowski, Marciniak, 1987)

✓ finite groups \( G \) with \( R(G) \neq 1 \)
  (Li, Parmenter, Sehgal, 1999)

✓ finite quasi-nilpotent groups,
  (Hertweck, Kimmerle, 2001)
Positive results

Theorem

(NP) holds for

✓ finite groups with normal Sylow 2-subgroups
  (Jackowski, Marciniak, 1987)

✓ finite groups $G$ with $R(G) \neq 1$
  (Li, Parmenter, Sehgal, 1999)

✓ finite quasi-nilpotent groups,
  finite 2-constrained groups $G$, where $G/O_2(G)$ has no chief factor of order 2
  (Hertweck, Kimmerle, 2001)
Positive results

Theorem

\((NP)\) holds for

✓ finite groups with normal Sylow 2-subgroups  
  (Jackowski, Marciniak, 1987)

✓ finite groups \(G\) with \(R(G) \neq 1\)  
  (Li, Parmenter, Sehgal, 1999)

✓ finite quasi-nilpotent groups,  
  finite 2-constrained groups \(G\), where \(G/O_2(G)\) has no chief factor of order 2  
  (Hertweck, Kimmerle, 2001)

✓ locally nilpotent groups,  
  (Jespers, Juriaans, de Miranda, Rogerio, 2002)
Positive results

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(NP) holds for</strong></td>
</tr>
<tr>
<td>✓ finite groups with normal Sylow 2-subgroups</td>
</tr>
<tr>
<td>✓ finite groups $G$ with $R(G) \neq 1$</td>
</tr>
<tr>
<td>✓ finite quasi-nilpotent groups, finite 2-constrained groups $G$, where $G/O_2(G)$ has no chief factor of order 2</td>
</tr>
<tr>
<td>✓ locally nilpotent groups, periodic groups with normal Sylow 2-subgroup</td>
</tr>
</tbody>
</table>

(Jackowski, Marciniak, 1987)
(Li, Parmenter, Sehgal, 1999)
(Hertweck, Kimmerle, 2001)
(Jespers, Juriaans, de Miranda, Rogerio, 2002)
Theorem (Hertweck, 1998)
There is a metabelian group of order \(2^{25} \cdot 97^2 = 315,713,650,688\) not satisfying (NP).
Theorem (Hertweck, 1998)

There is a metabelian group of order $2^{25} \cdot 97^2 = 315\ 713\ 650\ 688$ not satisfying (NP).
The normalizer problem for subgroups

\[ N_{U(RG)}(G) = G \cdot Z(U(RG)) \quad (NP) \]
The normalizer problem for subgroups

\[ N_{U(RG)}(G) = G \cdot Z(U(RG)) \quad \text{(NP)} \]

Let \( H \leq G \).
The normalizer problem for subgroups

\[ N_{U(RG)}(G) = G \cdot Z(U(RG)) \quad \text{(NP)} \]

\[ = N_G(G) \]

Let \( H \leq G \).
The normalizer problem for subgroups

\[ N_{U(RG)}(G) = G \cdot Z(U(RG)) \quad \text{(NP)} \]

\[ = N_G(G) \cdot C_{U(RG)}(G) \]

Let \( H \leq G \).
The normalizer problem for subgroups

\[ N_{U(RG)}(G) = G \cdot Z(U(RG)) \quad \text{(NP)} \]
\[ = N_G(G) \cdot C_{U(RG)}(G) \]

Let \( H \leq G \). We say that \( H \leq G \) has the normalizer property, if
The normalizer problem for subgroups

\[ \mathcal{N}_{U(RG)}(G) = G \cdot Z(U(RG)) \quad (NP) \]
\[ = \mathcal{N}_G(G) \cdot \mathcal{C}_{U(RG)}(G) \]

Let \( H \leq G \). We say that \( H \leq G \) has the normalizer property, if

\[ \mathcal{N}_{U(RG)}(H) = \mathcal{N}_G(H) \cdot \mathcal{C}_{U(RG)}(H), \]
The normalizer problem for subgroups

\[ N_{U(RG)}(G) = G \cdot Z(U(RG)) \]  \hspace{1cm} (NP)
\[ = N_G(G) \cdot C_{U(RG)}(G) \]

Let \( H \leq G \). We say that \( H \leq G \) has the normalizer property, if

\[ N_{U(RG)}(H) = N_G(H) \cdot C_{U(RG)}(H), \]

\((NP: H \leq G)\) for short.
The normalizer problem for subgroups

\[
N_{U(RG)}(G) = G \cdot Z(U(RG)) \hspace{1cm} (NP)
\]

\[
= N_G(G) \cdot C_{U(RG)}(G)
\]

Let \( H \leq G \). We say that \( H \leq G \) has the normalizer property, if

\[
N_{U(RG)}(H) = N_G(H) \cdot C_{U(RG)}(H),
\]

(NP: \( H \leq G \)) for short.

We say that \( G \) has the subgroup normalizer property, (SNP), if (NP: \( H \leq G \)) holds for all \( H \leq G \), i.e.

\[
\forall H \leq G : \quad N_{U(RG)}(H) = N_G(H) \cdot C_{U(RG)}(H),
\]
is again an automorphism problem
is again an automorphism problem

<table>
<thead>
<tr>
<th>Lemma</th>
</tr>
</thead>
<tbody>
<tr>
<td>((NP: H \leq G)) holds \iff \text{Aut}_{RG}(H) = \text{Aut}_G(H).</td>
</tr>
</tbody>
</table>
is again an automorphism problem

Lemma

$(NP: H \leq G)$ holds $\iff \text{Aut}_{RG}(H) = \text{Aut}_{G}(H)$.

Where

$$\text{Aut}_{RG}(H) = \{ \text{conj}(u) \in \text{Aut}(H) \mid u \in N_{U(RG)}(H) \} ,$$

$$\text{Aut}_{G}(H) = \{ \text{conj}(g) \in \text{Aut}(H) \mid g \in N_{G}(H) \} .$$
Results as a question on $H$

Remark (NP: $H \leq G$) holds, if $\text{Out}(H) = 1$.

Proposition

Let $H \leq G$, $H$ be cyclic. Then (NP: $H \leq G$) holds for arbitrary rings $R$.

Lemma (Coleman’s lemma, relative version)

Let $H \leq G$, $u \in \mathbb{N} U(R_G)(H)$ and $p$ a rational prime with $p \not\in U(R)$, then there exists $P \leq H$, such that $|H:P| < \infty$, $p \nmid |H:P|$ and $x \in \text{supp}(u) = \{g \in G | u^g \neq 0\}$ with $xu \in C U(R_G)(P)$. 
Results as a question on $H$

Remark

$(NP: H \leq G)$ holds, if $\text{Out}(H) = 1$. 
Results as a question on $H$

**Remark**

$(NP: H \leq G)$ holds, if $\text{Out}(H) = 1$.

**Proposition**

Let $H \leq G$, $H$ be cyclic. Then $(NP: H \leq G)$ holds for arbitrary rings $R$. 

 Lemma (Coleman's lemma, relative version)

Let $H \leq G$, $u \in \mathbb{N} \cup (\mathbb{R}G)^{(H)}$ and $p$ a rational prime with $p \notin \mathbb{U}(\mathbb{R}G)$, then there exists $P \leq H$, such that $|H:P| < \infty$, $p \nmid |H:P|$ and $x \in \text{supp}(u) = \{g \in G | u g \neq 0\}$ with $xu \in \mathbb{C} \cup (\mathbb{R}G)^{(P)}$. 

Remark

\((NP: H \leq G) \text{ holds, if } \text{Out}(H) = 1.\)

Proposition

Let \( H \leq G, H \) be cyclic. Then \((NP: H \leq G) \) holds for arbitrary rings \( R \).

Lemma (Coleman’s lemma, relative version)

Let \( H \leq G, u \in N_{U(RG)}(H) \) and \( p \) a rational prime with \( p \notin U(R) \), then there exists \( P \leq H \), such that \( |H : P| < \infty \), \( p \nmid |H : P| \) and \( x \in \text{supp}(u) = \{g \in G \mid u_g \neq 0\} \) with \( xu \in C_{U(RG)}(P) \).
Theorem

(SNP) holds for locally nilpotent torsion groups $G$ and $G$-adapted rings $R$. 

Theorem

(SNP) holds for locally nilpotent torsion groups $G$ and $G$-adapted rings $R$.

Proposition

(NP: $H \leq G$) holds for $G$ finitely generated nilpotent $H \leq G$ a torsion subgroup and $G$-adapted rings $R$. 

Results as a question on $G - 1$
Results as a question on $G - 1$

Theorem

(SNP) holds for locally nilpotent torsion groups $G$ and $G$-adapted rings $R$.

Proposition

(NP: $H \leq G$) holds for $G$ finitely generated nilpotent $H \leq G$ a torsion subgroup and $G$-adapted rings $R$.

Proposition

(SNP) holds for $G$ a finitely-generated torsion-free nilpotent groups and all rings $R$. 
Results as a question on $G$ - 2

Proposition

Let $G$ be finite metacyclic, $N \unlhd G$ and $G/N$ cyclic, such that $N$ is of prime order or $G/N$ is of prime order, then (SNP) holds for $G$.

Proposition

(SNP) holds for all groups of order at most 47.

(SNP) holds for all groups of order at most 659.
Let $G$ be finite metacyclic, $N \trianglelefteq G$ and $G/N$ cyclic, such that

- $N$ is of prime order or
- $G/N$ is of prime order,

then (SNP) holds for $G$. 

Proposition
Proposition

Let $G$ be finite metacyclic, $N \trianglelefteq G$ and $G/N$ cyclic, such that

- $N$ is of prime order or
- $G/N$ is of prime order,

then $(SNP)$ holds for $G$.

Proposition

$(SNP)$ holds for all groups of order at most 47.
$(SNP)$ holds for all groups of order at most 659.
Definition

Let $p$ be a prime and $X$ a finite group. An automorphism $\varphi \in \text{Aut}(X)$ is called $p$-central, if there is a Sylow $p$-subgroup $P$ of $X$ such that $\varphi|_P = \text{id}_P$. 

Proposition

Let $H \vartriangleleft G$ and $H$ be a finite simple group. Then $(\text{NP}: H \leq G)$ holds for $H$-adapted rings $R$.

Proposition

Let $H$ be a $p$-constrained group with $O_p(H) = 1$ for some prime $p$, and $H \vartriangleleft G$ or $H = N_G(P)$ for a $P \in \text{Syl}_p(G)$. Then $(\text{NP}: H \leq G)$ holds for rings $R$ with $p \not\in R\times$. 
Definition

Let $p$ be a prime and $X$ a finite group. An automorphism $\varphi \in \text{Aut}(X)$ is called $p$-central, if there is a Sylow $p$-subgroup $P$ of $X$ such that $\varphi|_P = \text{id}_P$.

Proposition

Let $H \trianglelefteq G$ and $H$ be a finite simple group. Then $(\text{NP}: H \leq G)$ holds for $H$-adapted rings $R$. 
Definition

Let $p$ be a prime and $X$ a finite group. An automorphism $\varphi \in \text{Aut}(X)$ is called $p$-central, if there is a Sylow $p$-subgroup $P$ of $X$ such that $\varphi|_P = \text{id}_P$.

Proposition

Let $H \unlhd G$ and $H$ be a finite simple group. Then (NP: $H \leq G$) holds for $H$-adapted rings $R$.

Proposition

Let $H$ be a $p$-constrained group with $O_p(H) = 1$ for some prime $p$, and $H \unlhd G$ or $H = N_G(P)$ for a $P \in \text{Syl}_p(G)$. Then (NP: $H \leq G$) holds for rings $R$ with $p \not\in R^\times$. 
Definition

The prime graph (or Gruenberg-Kegel graph) of a group $X$ is the undirected loop-free graph $\Gamma(X)$ with

- Vertices: primes $p$, s.t. there exists an element of order $p$ in $X$
- Edges: $p$ and $q$ joined iff there is an element of order $pq$ in $X$
Definition

The prime graph (or Gruenberg-Kegel graph) of a group $X$ is the undirected loop-free graph $\Gamma(X)$ with

- **Vertices:** primes $p$, s.t. there exists an element of order $p$ in $X$

Proposition (joint with W. Kimmerle)

Assume that $U$ is an isolated subgroup of the finite group $G$. Then

$$\Gamma(N_{Z(G)}(U)) = \Gamma(N_G(U)).$$
Definition

The prime graph (or Gruenberg-Kegel graph) of a group $X$ is the undirected loop-free graph $\Gamma(X)$ with

- **Vertices**: primes $p$, s.t. there exists an element of order $p$ in $X$
- **Edges**: $p$ and $q$ joined iff there is an element of order $pq$ in $X$
Definition

The prime graph (or Gruenberg-Kegel graph) of a group $X$ is the undirected loop-free graph $\Gamma(X)$ with

- Vertices: primes $p$, s.t. there exists an element of order $p$ in $X$
- Edges: $p$ and $q$ joined iff there is an element of order $pq$ in $X$

Proposition (joint with W. Kimmerle)

Assume that $U$ is an isolated subgroup of the finite group $G$. Then $\Gamma(N_{V(\mathbb{Z}_G)}(U)) = \Gamma(N_G(U))$. 
Thank you for your attention!