RATIONALITY OF GROUPS AND CENTERS OF INTEGRAL GROUP RINGS

Andreas Bächle
Groups St Andrews 2017
**Notation.**

\( G \) \textit{finite} group

\( \mathbb{Z}G \) integral group ring of \( G \)

\( U(\mathbb{Z}G) \) group of units of \( \mathbb{Z}G \)
CONTENTS

1. Rationality of Groups
2. Centers of Integral Group Rings
3. Solvable Groups
4. Frobenius Groups
5. References
**Definitions.** \( x \in G. \)

\[
x \text{ rational in } G \quad : \iff \quad \forall j \in \mathbb{Z} : \ x^j \sim x \\
\text{ with } (j, o(x)) = 1
\]

\[
x \text{ semi-rational in } G \quad : \iff \quad \exists m \in \mathbb{Z} \ \forall j \in \mathbb{Z} : \ x^j \sim x \quad \text{or} \quad x^j \sim x^m \\
\text{ with } (j, o(x)) = 1
\]

\[
x \text{ inverse semi-rational in } G \quad : \iff \quad \forall j \in \mathbb{Z} : \ x^j \sim x \quad \text{or} \quad x^j \sim x^{-1} \\
\text{ with } (j, o(x)) = 1
\]

\[G \text{ is called } \text{rational} \quad : \iff \quad \forall x \in G : x \text{ is rational in } G\]

etc.
For $\chi \in \text{Irr}(G)$, $x \in G$ set

$$Q(\chi) := Q(\{\chi(y) : y \in G\})$$

$$Q(x) := Q(\{\psi(x) : \psi \in \text{Irr}(G)\}).$$

- **G rational** \iff $\text{CT}(G) \in \mathbb{Q}^{h \times h}$
- **G semi-rational** \iff $\forall x \in G: [Q(x) : \mathbb{Q}] \leq 2$
- **G inverse semi-rational** \iff $\forall x \in G: Q(x) \subseteq \mathbb{Q}(\sqrt{-d_x}), d_x \in \mathbb{Z}_{\geq 0}$
\iff $\forall \chi \in \text{Irr}(G): Q(\chi) \subseteq \mathbb{Q}(\sqrt{-d_\chi}), d_\chi \in \mathbb{Z}_{\geq 0}$

$$\text{CT}(G) = \left( \begin{array}{ccc} \vdots & & \vdots \\ \vdash & \vdash & \vdash \\ \vdash & \vdash & \vdash \\ \vdash & \vdash & \vdash \\ \vdots & & \vdots \end{array} \right)$$

$$\text{CT}(G) = \left( \begin{array}{ccc} \vdash & & \vdash \\ \vdash & \vdash & \vdash \\ \vdash & \vdash & \vdash \\ \vdash & \vdash & \vdash \\ \vdots & & \vdots \end{array} \right)$$

$$\text{CT}(G) = \left( \begin{array}{ccc} \vdash & \vdash & \vdash \\ \vdash & \vdash & \vdash \\ \vdash & \vdash & \vdash \\ \vdash & \vdash & \vdash \\ \vdots & & \vdots \end{array} \right)$$
EXAMPLES.

- $S_n$ is rational.
- $P \in \text{Syl}_p(S_n)$.
  - $P$ rational $\iff p = 2$.
  - $P$ inverse semi-rational $\iff p \in \{2, 3\}$.
- $P \in \text{Syl}_p(\text{GL}(n, p^f))$.
  - $P$ rational $\iff p = 2$ and $n \leq 12$.
  - $P$ inverse semi-rational $\Rightarrow p = 2$ and $n \leq 24$ or $p = 3$ and $n \leq 18$.

DEFINITION.

$\pi(G) = \{p \text{ prime}: p \mid |G|\}$, the prime spectrum of $G$.

Then $|\pi(S_n)| \to \infty$ for $n \to \infty$. 
<table>
<thead>
<tr>
<th>G solvable</th>
<th>G rational</th>
<th>G inverse semi-rational</th>
<th>G semi-rational</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi(G) \subseteq {2, 3, 5}$</td>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
</tr>
<tr>
<td>Gow, 1976</td>
<td>$\Rightarrow$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Chilag-Dolfi, 2010</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>G</td>
<td>$ odd</td>
<td>$G = 1$</td>
</tr>
<tr>
<td>G simple</td>
<td>3 groups Feit-Seitz, 1989</td>
<td>25 groups</td>
<td>all $A_n$ + 41 groups Alavi-Daneshkhah, 2016</td>
</tr>
<tr>
<td>$</td>
<td>G</td>
<td>\leq 511$</td>
<td>$\approx 1%$</td>
</tr>
</tbody>
</table>
\[ \pm G \subseteq U(\mathbb{Z}G) \quad \text{– "trivial units"} \]
\[ \pm G = U(\mathbb{Z}G) \iff G \text{ abelian with } \exp G \mid 4 \text{ or } \exp G \mid 6 \quad \text{or} \]
\[ G \text{ Hamiltonian 2-group} \]
(Higman, 1940)

\[ \pm Z(G) \subseteq Z(U(\mathbb{Z}G)) \quad \text{– "trivial central units"} \]
\[ \pm Z(G) = Z(U(\mathbb{Z}G)) \iff G \text{ cut group} \]
(all central units trivial)

\[ \left[ U(\mathbb{Z}G) : \left\langle (\mathbb{Z}G)^1, Z(U(\mathbb{Z}G)) \right\rangle \right] < \infty \]

often up to f.i.
by "bicyclic units"
covered by "bicyclic units" &
"Bass units"
**Theorem (Ritter-Sehgal, et.al.)** For a finite group $G$ TFAE

1. $G$ is cut.
2. $\forall \chi \in \text{Irr}(G): \mathbb{Q}(\chi) \subseteq \mathbb{Q}(\sqrt{-d_\chi}), \ d_\chi \in \mathbb{Z}_{\geq 0}$.
3. $G$ is inverse semi-rational.
4. $K_1(\mathbb{Z}G)$ is finite.

In particular: $G$ cut $\Rightarrow$ $G/N$ cut for all $N \trianglelefteq G$. 
**Theorem (Bakshi-Maheshwary-Passi, 2016)** \( G \neq 1 \) cut-group

1. \( 2 \in \pi(G) \) or \( 3 \in \pi(G) \).
2. If \( G \) is nilpotent, then \( G \) is a \( \{2, 3\} \)-group.
3. If \( G \) is metacyclic, then \( G \) is in a list of 52 groups.

**Theorem (Maheshwary, 2016)** Let \( G \) be a solvable cut group.

1. If \( |G| \) is odd \( \implies \pi(G) \subseteq \{3, 7\} \) and all elements of \( G \) are of prime power order.
2. If \( |G| \) is even and all elements of \( G \) are of prime power order \( \implies \pi(G) \subseteq \{2, 3, 5, 7\} \).

**Theorem (B., 2017)** Let \( G \) be a solvable cut group.
Then \( \pi(G) \subseteq \{2, 3, 5, 7\} \).
**Theorem (B., 2017)** Let $G$ be a solvable cut group. Then $\pi(G) \subseteq \{2, 3, 5, 7\}$.

**Strategy of proof.**

- $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$ (Chillag-Dolfi).
- Let $G$ be a minimal counterexample, $V \triangleleft G$ minimal.
- Then $G \cong V \rtimes G/V$, $G/V$ is again cut.
- The $\mathbb{F}_{13}[G/V]$-module $V$ has the “12-eigenvalue property”.
- Derive restrictions on field of character values of $V$.
- By a result of Farias e Soares such a module cannot exist for a solvable group $G/V$. □
THEOREM (B., 2017). Let $K$ be a Frobenius complement.

(1) If $|K|$ is even ...
(2) If $|K|$ is odd ...
**Theorem (B., 2017).** Let $K$ be a Frobenius complement.

(1) If $|K|$ is even and the complement of a cut Frobenius group $G$, then $G$ is isomorphic to a group in the series on the left ($b, c, d \in \mathbb{Z}_{\geq 1}$) or one of the groups on the right.

(a) $C_3^b \rtimes C_2$  \hspace{1cm} (α) $C_5^2 \rtimes Q_8$
(b) $C_3^{2b} \rtimes C_4$  \hspace{1cm} (β) $C_7^2 \rtimes (C_3 \rtimes C_4)$
(c) $C_3^{2b} \rtimes Q_8$  \hspace{1cm} (γ) $C_7^2 \rtimes SL(2, 3)$
(d) $C_5^c \rtimes C_4$  \hspace{1cm} (δ) $C_7^2 \rtimes SL(2, 3)$
(e) $C_7^d \rtimes C_6$
(f) $C_7^{2d} \rtimes (Q_8 \times C_3)$

Conversely, for each of the above structure descriptions, there is a unique cut Frobenius group.

(2) If $|K|$ is odd ...
**Theorem (B., 2017).** Let $K$ be a Frobenius complement.

1. If $|K|$ is even ...

2. If $|K|$ is odd, then there is a cut Frobenius group $G$ if and only if $K \simeq C_3$ and the kernel $F$ is a group admitting a fixed-point free automorphism $\sigma$ of order 3 such that

   **(a)** $F$ is a cut 2-group.
   In particular, $|F| = 2^{2a}, a \in \mathbb{Z}_{\geq 1}$ and $F$ is an extension of an abelian group of exponent a divisor of 4 by an an abelian group of exponent a divisor of 4.

   **(b)** $F$ is an extension of an elementary abelian 7-group by an elementary abelian 7-group, $\exp F = 7$ and $\sigma$ fixes each cyclic subgroup of $F$. 
Strategy of proof.  $G$ cut Frobenius group with complement $K$.

- $K$ is also cut.
- Show that $K$ is solvable, so $\pi(G) \subseteq \{2, 3, 5, 7\}$.
- Determine possible structures of $P \in \text{Syl}_p(K)$.
- Determine possible structures of $K$.
- Use irreducible representations of these complements to describe structure of some $G$.
- Decide which subdirect products of the groups above are cut Frobenius groups.
References


