Matrix Wreath Products of Algebras.

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Wreath Product of Groups: \( G_2 \wr G_1 = G_2 \times \text{Fun}(G_2, G_1); g \in G_2, f: G_2 \to G_1 \)

\((g^{-1} f g)(g') = f(g g').\)

\(F\) a field; \(A, B\) are associative \(F\)-algebras.
Lin (B, A) = vector space of all linear transformations B → A.

\[ A \triangleright B = B + \text{Lin} (B, B \otimes_f A) \]

**Multiplication.** Let \( f, g \in \text{Lin} (B, B \otimes_f A) \).

Let \( b \in B \), \( g(b) = \sum_i b_i \otimes a_{i} \),

\[ f(b_i) = \sum_j b_{ij} \otimes a_{ij} \]  Then \( (fg)(b) = \sum b_{ij} \otimes a_{ij} a_i \).

\[
\begin{align*}
(fh)(b') &= f(bb') \\
(bf)(b') &= (b \otimes 1)f(b')
\end{align*}
\]

\[ f(b') = \sum_i b_i \otimes a_i \]

\[ (bf)(b') = \sum_i bb_i \otimes a_i \]
Thm. \( A \trianglelefteq B \) is an associative algebra.

We assume: \( B \cong 1, \) \( B \) is finitely generated, \( \dim_F B = 2^{2^a}. \)

\( \{ b_i \}_{i \in I} \) basis of \( B. \)

\( M_{I \times I} (A) = I \times I \) matrices having finitely many \( \neq 0 \) entries in each column.

Once we fix a basis:

\[ \text{Lin}(B, B \otimes F A) \cong M_{I \times I} (A) \]

\[ A \trianglelefteq B = B + M_{I \times I} (A). \]
$G_1, G_2$ groups, then

$$G_1 \rtimes G_2 \rightarrow (FG_1 \rtimes FG_2)^*$$

$f \in \text{Fun}(G_2, G_1)$, $f \Rightarrow$ diagonal matrix.

Inspired by:

(1) Jason Bell, Lance Small, Agata Smoktunowicz, Primitive algebraic algebras of polynomially bounded growth, 2012;

\( M_{\infty}(A) = \{ \text{matrices having finitely many } \neq 0 \text{ entries} \} \)

\( \gamma : B \to A \) \text{ linear map, } C_\gamma : b \mapsto 1 \otimes \gamma(b) \in B \otimes A. \)

All examples below: \textit{finitely generated algebras} \( \langle B, C_\gamma \rangle. \)

\textbf{Embedding Theorems.}

\textit{G. Higman, B. Neumann, H. Neumann}, 1949:

\textit{G countable group} \( \rightarrow \) \textit{finitely generated group.}

\textit{A. I. Malcev}, 1952: \textit{A associative,}

\( \dim_f A \leq \aleph_0 \rightarrow \textit{finitely generated associative algebra.} \)
A. I. Shirshov, 1958: A Lie algebra, \[ \dim_F L \leq \frac{1}{2}, \] \[ \iff \text{finitely generated Lie algebra}. \]

Can A be embedded as a (right, left) ideal? NO, but

Alahmadi, Alsulami, 2014: \[ A \rightarrow M_\infty (A) \supset B, \]
\[ B \text{ is finitely generated}. \]

A is \( \infty \)-embeddable as an ideal.

**Radical Algebras.**

A is \( \text{nil} \) if \( \forall a \in A \ a^n = 0. \)

\[ \text{Nil}(A) \subseteq \text{Jac}(A) \]
\[ \text{nil radical} \]

and Jacobson radicals of A.
Amitsur Problem: A finitely generated $\Rightarrow$

$\text{Jac}(A) = \text{Nil}(A)$.

Yes, if $F$ is uncountable.

K. Beidar, 1981: example.

J. Bell, 2003: more examples with finite Gelfand-Kirillov dimension.


Thm. (1) A countable dimensional Jacobson radical algebra $A \rightarrow$ finitely generated Jacobson radical algebra $A'$.
(2) If countable, $GKdim(A) \leq d \Rightarrow GKdim(A^d) \leq d + 6$.

Kurosh Problem: (1) Is a finitely generated nil algebra nilpotent?

(2) Is a finitely generated algebraic algebra finite dimensional?

Golod–Shafarevich, 1964: NO.

T. Lenagan - A. Smoktunowicz, 2007: If countable, $\exists$ a finitely generated infinite dimensional nil algebra $B$ of $GKdim(B) < \infty$. 
2012: $F$ countable, $E$ a finitely generated infinite dimensional nil algebra $B$ of $GKdim(B) \leq 3$. 

$\hat{B} = B + F1$ unital hull.

We consider: $A \in \hat{B}$ and the subalgebra $\langle B, Cx \rangle$.

**Nil Algebras.**

A is **stable nil** (resp. **stable algebraic**) if

$\forall n \geq 1 \quad M_n(A)$ in nil (resp. algebraic).

**Thm.** (1) A stable nil, $dim_p A \leq 3 \rightarrow$ finitely generated stable nil algebra $A'$,

(2) $F$ countable, $GKdim(A) \leq d \rightarrow GKdim(A') \leq d + 6$. 
Primitive Algebras.

A is (left) primitive if \( \exists \) an irreducible faithful module \( A \mathcal{M} \).

Kaplansky: \( \exists \) finitely generated infinite dimensional algebraic primitive algebra.

J. Bell - L. Small, 2002: YES.

J. Bell - L. Small - A. Smoktunowicz, 2012:

F countable, examples of finite G+K-dimension.

Theorem. (1) A stable algebraic primitive algebra, \( \dim F \leq 2 \), in \( \mathcal{M}_{\infty} \)-embeddable as a left ideal in a 2-generated algebraic primitive algebra;

(2) F countable, \( GKdim(A) \leq d \Rightarrow GKdim(A) \leq d+6. \)
Algebras of Subexponential Growth.

\[ A = \langle V \rangle , \dim_p V < \infty \]

\[ V^n = \text{Span} \{ v_1 \ldots v_k | v_i \in V, k \leq n \} \]

\[ V' \leq V^2 \leq \ldots , \ U V^n = A, \dim_p V^n < \infty \]

\[ g(n, V) = \dim_p V^n \] growth function of \( A \) that corresponds to \( V \).

\[ N = \{ 1, 2, \ldots \} ; \ f, g : N \to [1, \infty) \]

we say that \( f \preceq g \) (asymptotically less or equal to \( g \)) if \( \exists c \in N : \)

\[ f(n) \leq c g(cn) \ \forall n \geq 1 \]

If \( f \preceq g, g \preceq f \) then \( f \asymp g \), asymptotically equivalent.
If \( A = \langle V \rangle = \langle W \rangle \), \( \dim_f V < \infty \),
\( \dim_f W < \infty \) then \( g(V, n) \sim g(W, n) \)
\( G_A (n) = \text{class of equivalence} \).

A has polynomially bounded growth if
\( \exists \alpha > 0 : G_A (n) \leq n^\alpha \). Then
\( \inf \{ \alpha > 0 | G_A \leq n^\alpha \} = \text{SDK} \dim_f (A) \).

\( f(n) \) is subexponential if
\[
\frac{f(n)}{e^{\alpha n}} \to 0 \quad n \to \infty \quad \forall \alpha > 0.
\]

\( f(n) \) is intermediate if subexponential
and faster than any polynomial.

A be a not necessarily finitely generated algebra.

A is \textit{locally subexponential} if \( A \) finitely generated subalgebra \( B \trianglelefteq A \) \( g_B(n) \) is subexponential.

Growth of \( A \) is locally \( \leq f(n) \) if \( A \) finitely generated subalgebra \( B \trianglelefteq A \) \( g_B(n) \leq f(n) \).

L. Bartholdi - A. Erschler, 2014:

a countable locally subexponential group \( \rightarrow \) finitely generated group of subexponential growth.
\[ f, g : N \to [1, \infty) \quad f \preceq_w g \text{ weakly} \]

asymptotically less or equal than \( g \)

if \( \forall a \geq 0 \quad f \leq gn^a. \)

\[ f \succeq_w g \text{ if } f \preceq_w g, \quad g \succeq_w f. \]

A function \( h(n) \) is superlinear if

\[ \frac{h(n)}{n} \to \infty \text{ as } n \to \infty. \]

**Theorem.** Let \( f(n) \) be an increasing function; \( A \) is a countable dimensional algebra whose growth is locally \( \preceq_w f(n) \). Let \( h(n) \) be superlinear. Then \( A \to 2 \)-generated algebra \( G_B \preceq_w f(h(n))n^2. \)
Remark. This is a $M_{\infty}$-embedding as a left ideal.

Theorem. A countable dimensional algebra of locally subexponential growth $\rightarrow$ 2-generated algebra of subexponential growth.

Theorem (Alahmadi-Alsulami). Let char $F \neq 2$. Then a countable dimensional Lie algebra $A$ of locally subexponential growth $\rightarrow$ finitely generated Lie algebra of subexponential growth.

Question (Bell-Small-Smoktunowicz, 2012): is it true that $\forall$ sufficiently large $\alpha$ $\exists$ a finitely generated $A_\alpha$ with $GKdim (A_\alpha) = \alpha$?
**Theorem**: If countable. Then \( \forall \alpha \geq 8 \exists A_\alpha \text{ with } Gkdim (A_\alpha) = \alpha, \ A_\alpha \text{ is nil.} \)

*Bozho-Kraft*: \( \forall \alpha \geq 2 \quad \alpha = Gkdim (A_\alpha) \)

*U. Vishne*: \( \forall \alpha \geq 2 \quad \alpha = Gkdim (A_\alpha), A_\alpha \text{ is primitive.} \)

which functions are asymptotically equivalent to growth functions of groups, algebras ?

\((*)\) \( f(n) \) increases, \( f(m+n) \leq f(m) \cdot f(n) \).

**Theorem (Bartholdi-Erschler)**

Let \( \alpha \) be a positive root of \( x^3 - x^2 - 2x - 4 = 0 \), \( \alpha \approx 2.48 \); let \( \beta = \log_2 \alpha \approx 0.76 \).

If \( e^{n \beta} \leq f(n) \leq e^n \) then \( f(n) \sim \text{growth function.} \)
Question. Let \( g : \mathbb{N} \to \mathbb{N} \) be an increasing function, \( g(m+n) \leq g(m)g(n) \), and \( n^2 \leq g(n) \). Is \( g(n) \) asymptotically equivalent to the growth function of some finitely generated associative algebra?

Conjecture. For all sufficiently large functions \( g : \mathbb{N} \to \mathbb{N} \) the following assertions are equivalent:

1. \( g \sim g_A \), \( A \) a finitely generated associative algebra;
2. \( g \sim g_A \), \( A \) primitive;
3. \( g \sim g_A \), \( A \) is nil.