The algebraic structure of semi-brace

Paola Stefanelli

paola.stefanelli@unisalento.it

Università del Salento

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We introduce the *semi-brace*, an algebraic structure that allows us to obtain left non-degenerate solutions of the Yang-Baxter equation.

If $X$ is a set, a (set-theoretical) **solution** of the Yang-Baxter equation $r : X \times X \to X \times X$ is a map such that the well-known *braid equation*

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

is satisfied, where $r_1 = r \times \text{id}_X$ and $r_2 = \text{id}_X \times r$.

If $a, b \in X$, we denote $r(a, b) = (\lambda_a(b), \rho_b(a))$ where $\lambda_a, \rho_b$ are maps from $X$ into itself.

In particular, we say that $r$ is **left** (right, resp.) **non-degenerate** if $\lambda_a$ ($\rho_a$, resp.) is bijective, for every $a \in X$. Moreover, $r$ is **non-degenerate** if $r$ is both left and right non-degenerate.
In 2016, Guarnieri and Vendramin introduced a new algebraic structure, the skew braces, in order to obtain bijective solutions not necessarily involutive.

**Definition**

Let $B$ be a set with two operations $+$ and $\circ$ such that $(B, +)$ and $(B, \circ)$ are groups. We say that $(B, +, \circ)$ is a skew (left) brace if

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

holds for all $a, b, c \in B$, where $-a$ is the inverse of $a$ with respect to $+$. We may prove that the identity $0$ of $(B, +)$ is also the identity of $(B, \circ)$.

If the group $(B, +)$ is abelian, then $(B, +, \circ)$ is a brace, the algebraic structure introduced by Rump, in the reformulation provided by Cedó, Jespers and Okniński.

Clearly, every brace is a skew brace. Further, if $(B, +)$ is a group and we set $a \circ b := a + b$, for all $a, b \in B$, then $(B, +, \circ)$ is a skew brace, that we call zero skew brace. If $(B, +)$ is a non-abelian group then $B$ is a skew brace that is not a brace.
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Semi-braces

We introduced a generalization of skew braces.

**Definition (F. Catino, I. Colazzo, and P.S., J. Algebra, 2017)**

Let $B$ be a set with two operations $+$ and $\circ$ such that $(B, +)$ is a left cancellative semigroup and $(B, \circ)$ is a group. We say that $(B, +, \circ)$ is a (left) semi-brace if

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- $(B, +)$ is a group and, in particular, a left cancellative semigroup;
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$$a \circ b + a \circ (a^- + c) = a \circ b + a \circ a^- + a \circ c = a \circ b + 0 - a + a \circ c = a \circ b + a \circ c = a \circ (b + c)$$
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$\Rightarrow$ $(B, +)$ is a group and, in particular, a left cancellative semigroup;

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Semi-braces

We introduced a generalization of skew braces.

**Definition (F. Catino, I. Colazzo, and P.S., J. Algebra, 2017)**

Let $B$ be a set with two operations $+$ and $\circ$ such that $(B, +)$ is a left cancellative semigroup and $(B, \circ)$ is a group. We say that $(B, +, \circ)$ is a *(left) semi-brace* if

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P. Stefanelli (UniSalento)
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1. If \((E, \circ)\) is a group, then \((E, +, \circ)\), where \(a + b = b\), for all \(a, b \in E\) is a semi-brace. In fact,

   \(\begin{align*}
   \triangleright & \quad (E, +) \text{ is a left cancellative semigroup;} \\
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   \end{align*}\)

We call this semi-brace the **trivial semi-brace**.

2. If \((B, \circ)\) is a group and \(f\) is an endomorphism of \((B, \circ)\) such that \(f^2 = f\). Set

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   - if $a, b, c \in B$, then $a + (b + c) = a + (c \circ f(b)) = c \circ f(b) \circ f(a) = c \circ f(b) \circ f^2(a) = c \circ f(b \circ f(a)) = (a + b) + c$
   - $(B, +)$ is left cancellative;
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   - if $a, b, c \in B$, then

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The additive structure - I

Note that, if $B$ is a semi-brace and 0 is the identity of $(B, \circ)$, then 0 is a left identity (and, also, an idempotent) of $(B, +)$. In fact if $a \in B$, then

$$0 + a = 0 \circ (0 + a) = 0 \circ 0 + 0 \circ (0 + a) = 0 + 0 + a$$

and, by left cancellativity, we have that $a = 0 + a$.

Recall that a left cancellative semigroup $B$ is a right group if and only if for all $x, y \in B$ there exists $t \in B$ such that $x + t = y$.

If $B$ is a semi-brace, $x, y \in B$ and we set $t := x \circ (x^- + x^- \circ y)$, then

$$x + t = x + x \circ (x^- + x^- \circ y) = x \circ (0 + x^- + x^- \circ y) = x + y.$$

Hence, the additive structure $(B, +)$ is a right group.
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The additive structure - I

Note that, if $B$ is a semi-brace and $0$ is the identity of $(B, \circ)$, then $0$ is a left identity (and, also, an idempotent) of $(B, +)$. In fact if $a \in B$, then

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Moreover, it is well-known that if $B$ is a right group, $E$ is the set of idempotents, then $G_e := B + e$, for every $e \in E$, is a group and $B = G_e + E$.

In particular, if $B$ is a semi-brace and $E$ is the set of idempotents of $(B, +)$, then the identity $0$ of the group $(B, \circ)$ lies in $E$. Therefore $G := B + 0$ is a group with respect to the sum and

$$B = G + E.$$ 

In addition, we may prove that $(G, \circ)$ and $(E, \circ)$ are groups and so $(G, +, \circ)$ is a skew brace and $(E, +, \circ)$ is a trivial semi-brace.
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The additive structure – II

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Examples

- If \((E, +, \circ)\) is a trivial semi-brace, then the set of idempotents of \((E, +)\) is \(E\) and the group \(G = \{0\}\).

- If \((B, +, \circ)\) is the semi-brace where \(f : B \rightarrow B\) is an endomorphism of the group \((B, \circ)\), \(f^2 = f\) and \(a + b = b \circ f(a)\), for all \(a, b \in B\). The set of idempotents of \((B, +)\) is \(\text{ker} f\) and the group \(G := B + 0\) is \(\text{Im} f\). In fact,

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x \in E \iff x + x = x \iff x \circ f(x) = x \iff f(x) = 0 \iff x \in \text{ker} f.
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Let $B$ be a semi-brace, $E$ the set of idempotents of $(B, +)$, and $G := B + 0$.

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- Further, $B = G \circ E$. In fact, if $b \in B$, then there exist $g \in G$ and $e \in E$ such that $b = g + e$ and so

$$b = \left\{ \begin{array}{c} g \\ \in G \end{array} \right\} \circ \left\{ \begin{array}{c} g^- \circ (g + e) \\ \in E \end{array} \right\}.$$

In fact,

$$g^- \circ (g + e) + g^- \circ (g + e) = g^- \circ (g + e + e) - g^- \circ (g + e),$$

i.e., $g^- \circ (g + e) \in E$.

- Since $(G, \circ)$ and $(E, \circ)$ are groups, we have that $(B, \circ)$ is the matched product of the groups $(G, \circ)$ and $(E, \circ)$.  

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Guarnieri and Vendramin give the following definition of ideal for a skew brace.

**Definition**

Let $B$ be a skew brace. A subset $I$ of $B$ is said an ideal if

- $I$ is a normal subgroup of $(B, \circ)$;
- $I$ is a normal subgroup of $(B, +)$;
- $\lambda_a(I) \subseteq I$, for every $a \in B$, where $\lambda_a(b) := -a + a \circ b$, for all $a, b \in B$.

In particular, if $B$ is a brace, the second condition follows by the first and third ones.
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Ideals of a semi-brace - I

**Definition (F. Catino, I. Colazzo, P. S., J. Algebra, 2017)**

Let $B$ be a semi-brace, $E$ the set of idempotents of $(B, +)$, $G := B + 0$. We say that a subsemigroup $I$ of $(B, +)$ is an **ideal** if

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As we expect, $B$ and $\{0\}$ are ideals of $B$ that we call the **trivial ideals** of $B$. 
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Ideals and examples 

Additive and multiplicative structures 

Ideals and quotient structures 

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P. Stefanelli (UniSalento) 

*The algebraic structure of semi-brace*
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Where, if $a \in B$, $\lambda_a : B \rightarrow B$ and $\rho_a : B \rightarrow B$ are defined respectively by

$$\lambda_a(b) := a \circ (a^- + b) \quad \text{and} \quad \rho_a(b) := (b^- + a)^- \circ a$$

for every $b \in B$.

As we expect, $B$ and $\{0\}$ are ideals of $B$ that we call the **trivial ideals** of $B$. 
Definitions and examples
Additive and multiplicative structures
Ideals and quotient structures

Ideals of a semi-brace - I

Definition (F. Catino, I. Colazzo, P. S., J. Algebra, 2017)

Let $B$ be a semi-brace, $E$ the set of idempotents of $(B, +)$, $G := B + 0$. We say that a subsemigroup $I$ of $(B, +)$ is an ideal if

- $I$ is a normal subgroup of $(B, \circ)$;
- $I \cap G$ is a normal subgroup of $(G, +)$;
- $\lambda_g (e) \in I$, for all $g \in G$ and $e \in I \cap E$;
- $\rho_b (n) \in I$, for all $b \in B$ and $n \in I \cap G$.

Where, if $a \in B$, $\lambda_a : B \rightarrow B$ and $\rho_a : B \rightarrow B$ are defined respectively by

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- $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$.

Where, if $a \in B$, $\lambda_a : B \to B$ and $\rho_a : B \to B$ are defined respectively by

$$\lambda_a(b) := a ◦ (a^- + b) \quad \text{and} \quad \rho_a(b) := (b^- + a)^- ◦ a$$

for every $b \in B$.

As we expect, $B$ and $\{0\}$ are ideals of $B$ that we call the **trivial ideals** of $B$. 
Comparison between the two definitions of ideal - I

If $B$ is a skew brace and $I$ is an ideal of $B$, then $I$ is an ideal of $B$ reviewed as a semi-brace. In fact,

- $I$ is a normal subgroup of $(B, \circ)$;
- $I \cap G$ is a normal subgroup of $(G, +)$;
- $\lambda_g(e) \in I$, for all $g \in G$ and $e \in I \cap E$;
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\[
\rho_b(n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^- = (b^- \circ b - b^- + b^- \circ (-b + n^- + b))^- = (\lambda_b(-b + n^- + b))^- \in I.
\]
Comparison between the two definitions of ideal - I

If $B$ is a skew brace and $I$ is an ideal of $B$, then $I$ is an ideal of $B$ reviewed as a semi-brace. In fact,

- $I$ is a normal subgroup of $(B, \circ)$; ✔
- $I \cap G$ is a normal subgroup of $(G, +)$; ✔
- $\lambda_g (e) \in I$, for all $g \in G$ and $e \in I \cap E$; ✔
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\rho_b (n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^-
$$
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$$
Comparison between the two definitions of ideal – I

If $B$ is a skew brace and $I$ is an ideal of $B$, then $I$ is an ideal of $B$ reviewed as a semi-brace. In fact,

- $I$ is a normal subgroup of $(B, \circ)$; ✓
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- $\lambda_g (e) \in I$, for all $g \in G$ and $e \in I \cap E$; ✓
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\[ \rho_b (n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^- = (\lambda_b^- (-b + n^- + b))^- \in I. \]
Comparison between the two definitions of ideal - I

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- $\lambda_g(e) \in I$, for all $g \in G$ and $e \in I \cap E$; ✓
- $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$.

$$
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= (\lambda_b(-b + n^- + b))^\perp \in I.
$$
Comparison between the two definitions of ideal - I

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\[
\rho_b (n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^-
\]
\[
= (b^- \circ b - b^- + b^- \circ (-b + n^- + b))^- = (\lambda_b (b + n^- + b))^- \in I.
\]
Comparison between the two definitions of ideal – I

If $B$ is a skew brace and $I$ is an ideal of $B$, then $I$ is an ideal of $B$ reviewed as a semi-brace. In fact,

- $I$ is a normal subgroup of $(B, \circ)$; ✔
- $I \cap G$ is a normal subgroup of $(G, +)$; ✔
- $\lambda_g(e) \in I$, for all $g \in G$ and $e \in I \cap E$;
- $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$.

$$
\rho_b(n) = (n^− + b)^− \circ b = (b^− \circ (n^− + b))^− = (b^− \circ (b − b + n^− + b))^− = (\lambda_b(−b + n^− + b))^− \in I.
$$

P. Stefanelli (UniSalento)
Comparison between the two definitions of ideal - I

If $B$ is a skew brace and $I$ is an ideal of $B$, then $I$ is an ideal of $B$ reviewed as a semi-brace. In fact,

- $I$ is a normal subgroup of $(B, \circ)$; ✓
- $I \cap G$ is a normal subgroup of $(G, +)$; ✓
- $\lambda_g(e) \in I$, for all $g \in G$ and $e \in I \cap E$; $\lambda_g(0) = -g + g \circ 0 = 0 \in I$
- $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$.
  
  $\rho_b(n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^- = (b^- \circ b - b^- + b^- \circ (-b + n^- + b))^- = (\lambda_b(-b + n^- + b))^- \in I$.
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- $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$. 

$$
\rho_b(n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^-
$$

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- $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$. ✓

\[
\rho_b(n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^-
\]

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Comparison between the two definitions of ideal - I

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- $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$. ✔

$$
\rho_b(n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b^n^- + b))^-
$$
$$
= (b^- \circ b - b^- + b^- \circ (-b + n^- + b))^- = (\lambda_{b^-}(-b + n^- + b))^- \in I.
$$
If $B$ is a skew brace and $I$ is an ideal of $B$, then $I$ is an ideal of $B$ reviewed as a semi-brace. In fact,

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Comparison between the two definitions of ideal – I

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\[
\rho_b(n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^-
\]
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\]
\[
= (\lambda_{b^-}(-b + n^- + b))^- \in I.
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Comparison between the two definitions of ideal - I

If $B$ is a skew brace and $I$ is an ideal of $B$, then $I$ is an ideal of $B$ reviewed as a semi-brace. In fact,

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- $\lambda_g(e) \in I$, for all $g \in G$ and $e \in I \cap E$; ✓
- $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$. ✓

$$\rho_b(n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^-
= (b^- \circ b - b^- + b^- \circ (-b + n^- + b))^-
= (\lambda_b(\underbrace{-b + n^- + b})^-) \in I.$$
Comparison between the two definitions of ideal - I

If $B$ is a skew brace and $I$ is an ideal of $B$, then $I$ is an ideal of $B$ reviewed as a semi-brace. In fact,

- $I$ is a normal subgroup of $(B, \circ); \checkmark$
- $I \cap G$ is a normal subgroup of $(G, +); \checkmark$
- $\lambda_g (e) \in I$, for all $g \in G$ and $e \in I \cap E; \checkmark$
- $\rho_b (n) \in I$, for all $b \in B$ and $n \in I \cap G$. \checkmark

$$
\rho_b (n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^-
= (b^- \circ b - b^- + b^- \circ (-b + n^- + b))^-
= (\lambda_b (-b + n^- + b))^\in I. \checkmark
$$
Comparison between the two definitions of ideal - I

If $B$ is a skew brace and $I$ is an ideal of $B$, then $I$ is an ideal of $B$ reviewed as a semi-brace. In fact,

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- $I \cap G$ is a normal subgroup of $(G, +)$; ✔
- $\lambda_g (e) \in I$, for all $g \in G$ and $e \in I \cap E$; ✔
- $\rho_b (n) \in I$, for all $b \in B$ and $n \in I \cap G$. ✔

\[
\rho_b (n) = (n^- + b)^- \circ b = (b^- \circ (n^- + b))^- = (b^- \circ (b - b + n^- + b))^- = (b^- \circ b - b^- + b^- \circ (-b + n^- + b))^- = (\lambda_b (-b + n^- + b))^- \in I.
\]
Conversely, if $B$ a skew brace and $I$ is an ideal of $B$ reviewed as a semi-brace, then $I$ is an ideal of the skew brace $B$. In fact,

- $I$ is a normal subgroup of $(B, \circ)$; ✓
- $I$ is a normal subgroup of $(B, +)$; ✓
- $\lambda_a(I) \subseteq I$, for every $a \in B$. ✓

\[
\lambda_a(x) = a \circ (a^- + x) = (a^- + x)^- \circ a^- = ((a^- + x - a^- + a^-) \circ a^-)^-
\]
\[
= (\rho_{a^-}(a^- + x - a^-))^- \in I.
\]
Conversely, if $B$ a skew brace and $I$ is an ideal of $B$ reviewed as a semi-brace, then $I$ is an ideal of the skew brace $B$. In fact,

- $I$ is a normal subgroup of $(B, \circ)$; ✔
- $I$ is a normal subgroup of $(B, +)$; ✔
- $\lambda_a(I) \subseteq I$, for every $a \in B$. ✔

$$
\lambda_a(x) = a \circ (a^- + x) = ((a^- + x) \circ a^-)^- = ((a^- + x - a^- + a^-) \circ a^-)^- = \left(\rho_a^- (a^- + x - a^-)\right)^- \in I.
$$
Comparison between the two definitions of ideal – II

Conversely, if $B$ a skew brace and $I$ is an ideal of $B$ reviewed as a semi-brace, then $I$ is an ideal of the skew brace $B$. In fact,

- $I$ is a normal subgroup of $(B, \circ)$; ✔
- $I$ is a normal subgroup of $(B, +)$; ✔
- $\lambda_a(I) \subseteq I$, for every $a \in B$. ✔

\[
\lambda_a(x) = a \circ (a^- + x) = \left((a^- + x)^- \circ a^-\right)^- = \left((a^- + x - a^- + a^-) \circ a^-\right)^- = \left(\rho_a^{-1}(a^- + x - a^-)\right)^- \in I.
\]
Conversely, if $B$ a skew brace and $I$ is an ideal of $B$ reviewed as a semi-brace, then $I$ is an ideal of the skew brace $B$. In fact,

- $I$ is a normal subgroup of $(B, \circ)$; ✓
- $I$ is a normal subgroup of $(B, +)$; ✓
- $\lambda_a(I) \subseteq I$, for every $a \in B$. ✓

\[
\lambda_a(x) = a \circ (a^- + x) = \left((a^- + x)^{\circ} \circ a^-\right)^- = ((a^- + x - a^- + a^-) \circ a^-)^- = \left(\rho_{a^-}(a^- + x - a^-)\right)^- \in I.
\]
Conversely, if $B$ a skew brace and $I$ is an ideal of $B$ reviewed as a semi-brace, then $I$ is an ideal of the skew brace $B$. In fact,

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- $\lambda_a(I) \subseteq I$, for every $a \in B$. ✓

\[
\lambda_a(x) = a \circ (a^- + x) = \left( (a^- + x) \circ a^- \right)^- = \left( (a^- + x - a^- + a^-) \circ a^- \right) = \left( \rho_a (a^- + x - a^-) \right)^- \in I.
\]
Comparison between the two definitions of ideal - II

Conversely, if \( B \) a skew brace and \( I \) is an ideal of \( B \) reviewed as a semi-brace, then \( I \) is an ideal of the skew brace \( B \). In fact,

- \( I \) is a normal subgroup of \( (B, \circ) \); ✔
- \( I \) is a normal subgroup of \( (B, +) \); ✔
- \( \lambda_a(I) \subseteq I \), for every \( a \in B \). ✔

\[
\lambda_a(x) = a \circ (a^- + x) = ((a^- + x)^- \circ a^-)^- = ((a^- + x - a^- + a^-) \circ a^-)^- = (\rho_a^- (a^- + x - a^-))^- \in I.
\]
Conversely, if $B$ a skew brace and $I$ is an ideal of $B$ reviewed as a semi-brace, then $I$ is an ideal of the skew brace $B$. In fact,

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\[
\lambda_a(x) = a \circ (a^- + x) = \left((a^- + x)^- \circ a^-\right)^- = \left((a^- + x - a^- + a^-) \circ a^-\right)^- = \left(\rho_a\left((a^- + x - a^-)\right)\right)^- \in I.
\]
Comparison between the two definitions of ideal - II

Conversely, if $B$ a skew brace and $I$ is an ideal of $B$ reviewed as a semi-brace, then $I$ is an ideal of the skew brace $B$. In fact,

- $I$ is a normal subgroup of $(B, \circ)$;
- $I$ is a normal subgroup of $(B, +)$;
- $\lambda_a(I) \subseteq I$, for every $a \in B$. 

\[
\lambda_a(x) = a \circ (a^- + x) = \left( (a^- + x)^- \circ a^- \right)^- = \left( (a^- + x - a^- + a^-) \circ a^- \right)^- \\
= (\rho_{a^-}(a^- + x - a^-))^- \in I.
\]
Comparison between the two definitions of ideal \(- II\)

Conversely, if \(B\) a skew brace and \(I\) is an ideal of \(B\) reviewed as a semi-brace, then \(I\) is an ideal of the skew brace \(B\). In fact,

\begin{itemize}
  \item \(I\) is a normal subgroup of \((B, \circ)\);
  \item \(I\) is a normal subgroup of \((B, +)\);  
  \item \(\lambda_a(I) \subseteq I\), for every \(a \in B\).
\end{itemize}

\[
\lambda_a(x) = a \circ (a^- + x) = \left( (a^- + x)^- \circ a^- \right)^- = \left( (a^- + x - a^- + a^-) \circ a^- \right)^- \\
= (\rho_{a^-}(a^- + x - a^-))^- \in I.
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Comparison between the two definitions of ideal - II

Conversely, if $B$ a skew brace and $I$ is an ideal of $B$ reviewed as a semi-brace, then $I$ is an ideal of the skew brace $B$. In fact,

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- $\lambda_a(I) \subseteq I$, for every $a \in B$. ✓

\[
\lambda_a(x) = a \circ (a^- + x) = \left((a^- + x)^- \circ a^\circ\right)^- = \left((a^- + x - a^- + a^-) \circ a^-\right)^- = (\rho_a^- (a^- + x - a^-))^- \in I.
\]
Comparison between the two definitions of ideal - II

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- $\lambda_a(I) \subseteq I$, for every $a \in B$. ✓

\[
\lambda_a(x) = a \circ (a^– + x) = \left((a^– + x)^– \circ a^-\right)^– = \left((a^- + x - a^- + a^-) \circ a^-\right)^– \\
= (\rho_{a^-}(a^– + x - a^-))^– \in I.
\]
Remark

In the case of a semi-brace that is not a skew brace, there is a difference with respect to the ideal of a skew brace.

Let \((B, \circ)\) be a group that is not simple, reviewed as trivial semi-brace. If \(I\) is a non-trivial normal subgroup of \((B, \circ)\), then \(I\) is an ideal. In fact,

- \(I\) is a normal subgroup of \((B, \circ)\);
- \(I \cap G = \{0\}\) is a normal subgroup of \((G, +)\);
- \(\lambda_0(e) = 0 \circ (0 + e) = e \in I\), for every \(e \in I \cap E = I\);
- \(\rho_b(0) = (0 + b)^- \circ b = b^- \circ b = 0 \in I\), for every \(b \in B\).

But if \(a \in B \setminus I\), then

\[
\lambda_a(0) = a \circ (a^- + 0) = a \circ 0 = a \notin I,
\]

i.e., \(I\) is not \(\lambda_a\)-invariant, nevertheless \(I\) is an ideal of \(B\).
Remark

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- \(\lambda_0(e) = 0 \circ (0 + e) = e \in I\), for every \(e \in I \cap E = I\);
- \(\rho_b(0) = (0 + b)^- \circ b = b^- \circ b = 0 \in I\), for every \(b \in B\).

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- \(I \cap G = \{0\}\) is a normal subgroup of \((G, +)\);
- \(\lambda_0(e) = 0 \circ (0 + e) = e \in I\), for every \(e \in I \cap E = I\);
- \(\rho_b(0) = (0 + b)^- \circ b = b^- \circ b = 0 \in I\), for every \(b \in B\).

But if \(a \in B \setminus I\), then

\[
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Ideals of a semi-brace - II

**Proposition (F. Catino, I. Colazzo, P.S., J. Algebra, 2017)**

If $B$ is a semi-brace and $I$ is an ideal of $B$, then the relation $\sim_I$ on $B$ given by

$$\forall x, y \in B, \quad x \sim_I y \iff y^\circ x \in I$$

is a congruence of $B$.

Further, if $B$ is a semi-brace and $I$ is an ideal, then the quotient structure $B/I$ of $B$ with respect to the relation $\sim_I$ is a right group with respect to the sum.

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The socle

Guarnieri and Vendramin introduced the socle for skew braces, as generalization of that classical for braces.

**Definition**

Let $B$ be a skew brace. Then the ideal defined by

$$S(B) := \{ a \mid a \in B, \forall b \in B \quad a \circ b = a + b, \ b + b \circ a = b \circ a + b \}$$

is said the socle of $B$.

We may generalize this definition for semi-braces in the following way:

**Definition (F. Catino, I. Colazzo, P.S., J. Algebra, 2017)**

If $B$ is a semi-brace, $0$ is the identity of $(B, \circ)$ and $G := B + 0$, then we call the set given by

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We may generalize this definition for semi-braces in the following way:

**Definition (F. Catino, I. Colazzo, P.S., J. Algebra, 2017)**

If $B$ is a semi-brace, 0 is the identity of $(B, \circ)$ and $G := B + 0$, then we call the set given by

$$\text{Soc}(B) = \{a \mid a \in G, \forall b \in B \quad a \circ b = a + b, \quad -a + b + a = b + 0\}.$$ 

the **socle** of the semi-brace $B$.

If $B$ is a skew brace, then $S(B) = \text{Soc}(B)$. 
The socle

Guarnieri and Vendramin introduced the socle for skew braces, as generalization of that classical for braces.

**Definition**

Let $B$ be a skew brace. Then the ideal defined by

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If $B$ is a skew brace, then $S(B) = \text{Soc}(B)$. 
Thanks for your attention!