Group identities for unitary units of group rings

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Groups, Rings and the Yang-Baxter equation
Let $\langle x_1, x_2, \ldots \rangle$ be the free group on a countable infinitude of generators.

**Definition**

A subset $S$ of a group $G$ satisfies a *group identity* (and write $S$ is GI) if there exists a non-trivial reduced word $w(x_1, \ldots, x_n) \in \langle x_1, x_2, \ldots \rangle$ such that $w(g_1, \ldots, g_n) = 1$ for all $g_i \in S$. 
Examples

Let us define

\[(x_1, x_2) := x_1^{-1} x_2^{-1} x_1 x_2\]

and recursively

\[(x_1, \ldots, x_{n+1}) := ((x_1, \ldots, x_n), x_{n+1}).\]
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A group $G$ is *abelian* if it satisfies $(x_1, x_2) = 1$.
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- A group \(G\) is **abelian** if it satisfies \((x_1, x_2) = 1\)
- \(G\) is **nilpotent** if it satisfies \((x_1, x_2, \ldots, x_n) = 1\) for some \(n \geq 2\)
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- \(G\) is **nilpotent** if it satisfies \((x_1, x_2, \ldots, x_n) = 1\) for some \(n \geq 2\)
- \(G\) is **bounded Engel** if it satisfies \((x_1, x_2, \ldots, x_2) = 1\) for some \(n \geq 1\)
Hartley’s Conjecture

Let $F$ be a field and $G$ a torsion group.

$\mathcal{U}(FG)$ is GI $\implies$ $FG$ is PI
Hartley’s Conjecture

Let $F$ be a field and $G$ a torsion group.

$$\mathcal{U}(FG) \text{ is GI} \implies FG \text{ is PI}$$

Let $F\langle X \rangle$ be the free associative algebra generated by a countable set $X := \{x_1, x_2, \ldots \}$ over $F$.

**Definition**

A subset $S$ of an $F$-algebra $A$ is said to satisfy a *polynomial identity* (and write $S$ is PI) if there exists $0 \neq f(x_1, \ldots, x_n) \in F\langle X \rangle$ such that $f(a_1, \ldots, a_n) = 0$ for all $a_i \in S$. 
Solution of Hartley’s Conjecture

- Giambruno-Jespers-Valenti (1994): $\text{char } F = 0$ or $F$ infinite, $\text{char } F = p \geq 2$ and $P = 1$
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Characterization of when $\mathcal{U}(FG)$ is GI

- Passman (1997): $F$ infinite and $G$ torsion
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Characterization of when \( U(FG) \) is GI

- Passman (1997): \( F \) infinite and \( G \) torsion
- Liu-Passman (1999): \( F \) finite and \( G \) torsion
- Giambruno-Sehgal-Valenti (2000): \( G \) non-torsion
Classical and $F$-linear involutions

Let $G$ be a group endowed with an involution $\star$. Let us consider the $F$-linear extension of $\star$ to $FG$ setting

$$\left(\sum_{g \in G} a_g g\right)^\star := \sum_{g \in G} a_g g^\star.$$ 

This extension, which we denote again by $\star$, is an involution of $FG$ which fixes the ground field $F$ elementwise.
Classical and $F$-linear involutions

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$$

This extension, which we denote again by $\star$, is an involution of $FG$ which fixes the ground field $F$ elementwise. As is well-known, any group $G$ has a natural involution which is given by the map $\star : g \mapsto g^{-1}$.

**Definition**

Let $FG$ be the group algebra of a group $G$ over a field $F$. If $G$ is endowed with an involution $\star$, its linear extension to the group algebra $FG$ is called a $F$-linear involution of $FG$. In particular, if $\star = \ast$ the induced involution is called the classical involution.
Let us consider

$$\mathcal{U}^+(FG) := \{ x \mid x \in \mathcal{U}(FG) \quad x = x^* \},$$

$$Un(FG) := \{ x \mid x \in FG \quad xx^* = x^*x = 1 \}.$$

$Un(FG)$ is a subgroup of $\mathcal{U}(FG)$, whereas $\mathcal{U}^+(FG)$ is a subset of $\mathcal{U}(FG)$.

**Definition**

Let $FG$ be the group algebra of a group $G$ over a field $F$ endowed with an $F$-linear involution. The elements of $\mathcal{U}^+(FG)$ are called the *symmetric units* of $FG$ (with respect to $*$) and those of $Un(FG)$ are called the *unitary units* of $FG$. 
Symmetric and unitary units

Let us consider

\[ U^+(FG) := \{ x \mid x \in U(FG) \quad x = x^* \}, \]

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\( Un(FG) \) is a subgroup of \( U(FG) \), whereas \( U^+(FG) \) is a subset of \( U(FG) \).

**Definition**

Let \( FG \) be the group algebra of a group \( G \) over a field \( F \) endowed with an \( F \)-linear involution. The elements of \( U^+(FG) \) are called the **symmetric units** of \( FG \) (with respect to \( * \)) and those of \( Un(FG) \) are called the **unitary units** of \( FG \).
We want to determine if we can decide the structure of $G$ by imposing constraints upon subsets of the unit group $\mathcal{U}(FG)$. 
Constraints on subsets of $\mathcal{U}(FG)$

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**Main Question**

$Un(FG) / \mathcal{U}^+(FG)$ satisfies $\mathcal{P}$ $\implies$ $\mathcal{U}(FG)$ satisfies $\mathcal{P}$
We want to determine if we can decide the structure of $G$ by imposing constraints upon subsets of the unit group $\mathcal{U}(FG)$.

**Main Question**

$Un(FG) / \mathcal{U}^+(FG)$ satisfies $\mathcal{P} \implies \mathcal{U}(FG)$ satisfies $\mathcal{P}$ or $G$ is ....
Introduction and Motivations
Group Identities for $U^+(FG)$

When $U^+(FG)$ is GI
Group Identities for $Un(FG)$

Special Group Identities

When $U^+(FG)$ is GI

Theorem [Giambruno-Polcino Milies-Sehgal, 2009]

Let $FG$ be the group algebra of a torsion group $G$ over an infinite field $F$ of characteristic $p \neq 2$ endowed with a $F$-linear involution. Then $U^+(FG)$ is GI if, and only if,

1. $FG$ is semiprime and $G$ is either abelian or an SLC-group,
2. $FG$ is not semiprime, $P$ is a normal subgroup of $G$, $G$ has a $p$-abelian normal subgroup of finite index and either $G'$ is a $p$-group of bounded exponent or $G/P$ is an SLC-group and $G$ contains a normal $\star$-invariant $p$-subgroup $B$ of bounded exponent such that $P/B$ is central in $G/P$ and the induced involution acts as the identity on $P/B$.

Giambruno-Sehgal-Valenti (1998) answered the problem for the classical involution
When $\mathcal{U}^+(FG)$ is GI

**Theorem [Giambruno-Polcino Milies-Sehgal, 2009]**

Let $FG$ be the group algebra of a torsion group $G$ over an infinite field $F$ of characteristic $p \neq 2$ endowed with a $F$-linear involution. Then $\mathcal{U}^+(FG)$ is GI if, and only if,

(a) $FG$ is semiprime and $G$ is either abelian or an SLC-group, or

(b) $FG$ is not semiprime, $P$ is a normal subgroup of $G$, $G$ has a $p$-abelian normal subgroup of finite index and either
   - $G'$ is a $p$-group of bounded exponent or
   - $G/P$ is an SLC-group and $G$ contains a normal $\star$-invariant $p$-subgroup $B$ of bounded exponent such that $P/B$ is central in $G/P$ and the induced involution acts as the identity on $P/B$. 

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**Theorem [Giambruno-Polcino Milies-Sehgal, 2009]**

Let \( FG \) be the group algebra of a torsion group \( G \) over an infinite field \( F \) of characteristic \( p \neq 2 \) endowed with a \( F \)-linear involution. Then \( \mathcal{U}^+(FG) \) is GI if, and only if,

(a) \( FG \) is semiprime and \( G \) is either abelian or an SLC-group, or

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   - \( G' \) is a \( p \)-group of bounded exponent or
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Giambruno-Sehgal-Valenti (1998) answered the problem for the classical involution
A group $G$ is called an **LC-group** (that is, it has the "lack of commutativity" property) if it is not abelian, but, whenever $g, h \in G$ and $gh = hg$, then at least one of $\{g, h, gh\}$ is central. A group $G$ is an LC-group with a unique nonidentity commutator (which must, obviously, have order 2) if, and only if, $G/\zeta(G) \cong C_2 \times C_2$.

**Definition**

A group $G$ endowed with an involution $\ast$ is said to be a **special LC-group**, or **SLC-group**, if it is an LC-group, it has a unique nonidentity commutator $z$ and, for all $g \in G$, we have $g^\ast = g$ if $g \in \zeta(G)$ and, otherwise, $g^\ast = zg$. 


The non-torsion case

For infinite fields the question was studied by

- Sehgal-Valenti (2006): * is the classical involution
The non-torsion case

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- Sehgal-Valenti (2006): $\ast$ is the classical involution
- Giambruno-Polcino Milies-Sehgal (2017): in the more general framework of $\ast$-group identities
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Special group identities

**Classical involution**

- **Nilpotency**: Lee (2003) and Lee-Polcino Milies-Sehgal (2007)
Special group identities

Classical involution

- **Nilpotency**: Lee (2003) and Lee-Polcino Milies-Sehgal (2007)
- **Bounded Engel**: Lee-S. (2010)
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- **Nilpotency**: Lee (2003) and Lee-Polcino Milies-Sehgal (2007)
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Classical involution
- **Nilpotency**: Lee (2003) and Lee-Polcino Milies-Sehgal (2007)
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Theorem [Lee-Sehgal-S., 2010]
Let $F$ be an infinite field of characteristic $p > 2$ and $G$ a torsion group having an involution $\star$, and let $FG$ have the induced involution. Suppose that $\mathcal{U}(FG)$ is not nilpotent. Then $\mathcal{U}^+(FG)$ is nilpotent if, and only if, $G$ is nilpotent and $G$ has a finite normal $\star$-invariant $p$-subgroup $N$ such that $G/N$ is an $SLC$-group.
A result for finite groups

Theorem [Goncalves-Passman, 2001]

Let $FG$ be the group algebra of a finite group $G$ over a non-absolute field $F$ of characteristic $p \neq 2$ endowed with the classical involution. The unitary unit subgroup $Un(FG)$ contains no non-abelian free subgroup if, and only if,

(i) $G$ has a normal Sylow $p$-subgroup $P$ (by convention $P = 1$ if $p = 0$).

(ii) Either $\overline{G} := G/P$ is abelian or it has an abelian subgroup $\overline{A}$ of index 2. Furthermore, if the latter occurs, then either $\overline{G} = \overline{A} \rtimes \langle y \rangle$ is dihedral, or $\overline{A}$ is an elementary abelian 2-group.
**Theorem [Giambruno-Polcino Milies, 2003]**

Let $GF$ be the group algebra of a group $G$ over a field $F$ of characteristic 0 endowed with the classical involution. Suppose that $Un(FG)$ satisfies a group identity which is 2-free. Then the set $T$ of torsion elements of $G$ is a subgroup and one of the following conditions holds:

(i) $T$ is abelian.

(ii) $A := \{ g \mid g \in T \ o(g) \neq 2 \}$ is a normal abelian subgroup of $G$ and $(T \setminus A)^2 = 1$.

(iii) $T$ contains an elementary abelian 2-subgroup $B$ such that $[T : B] = 2$.

Conversely, if $G = T$ is a torsion group and $G$ satisfies one of the above conditions, then $Un(FG)$ is GI.
Some partial results for $F$-linear involutions

- Broche-Dooms-Ruiz (2009): $F$ non-absolute field of characteristic different from 2 and
  - $FG$ is regular
  - $G$ is locally finite and the prime radical of $FG$ is nilpotent
Nilpotency of $\mathcal{U}(FG)$

Theorem [Khripta, 1972]

Let $FG$ be the group algebra of a group $G$ over a field $F$ of characteristic $p > 0$ such that $FG$ is modular. Then $\mathcal{U}(FG)$ is nilpotent if, and only if, $G$ is nilpotent and $p$-abelian.
Nilpotency of $\mathcal{U}(FG)$

**Theorem [Khripta, 1972]**

Let $FG$ be the group algebra of a group $G$ over a field $F$ of characteristic $p > 0$ such that $FG$ is modular. Then $\mathcal{U}(FG)$ is nilpotent if, and only if, $G$ is nilpotent and $p$-abelian.
The non-modular case

Theorem [Fisher-Parmenter-Sehgal, 1976 & Khripta, 1971]

Let $FG$ be the group algebra of a group $G$ over a field $F$ of characteristic $p \geq 0$ such that, if $p > 0$, $FG$ is non-modular. Then the following are equivalent:

(i) $U(FG)$ is bounded Engel and solvable;

(ii) $G$ is bounded Engel and solvable, the torsion elements of $G$ form an abelian (normal) subgroup $T$ and either:

(a) $T$ is central in $G$ or

(b) $|F| = p$ is a Mersenne prime, $T^{p^2 - 1} = 1$ and for every $h \in T$ and $g \in G$ we have $h^g = h$ or $h^p$;

(iii) $U(FG)$ is nilpotent.
The modular case for unitary units

**Theorem [Lee-Sehgal-S., 2017]**

Let $FG$ be the group algebra of a group $G$ over an infinite field $F$ of characteristic $p > 2$ endowed with the classical involution, such that $FG$ is modular. Then the following are equivalent:

(i) $Un(FG)$ is bounded Engel and solvable;

(ii) $U(FG)$ is nilpotent;

(iii) $G$ is nilpotent and $p$-abelian.
Assume that $G$ has no 2-elements.
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- $G$ is locally finite.
The non-modular case for torsion groups

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- Assume also that $p \neq 0$ because of [GPM].
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Let $e_1, \ldots, e_k$ be the primitive central idempotents of $FG$. Then each $FGe_i = M_{n_i}(K_i)$ for some finite field $K_i$. 
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- $e_i^* \neq e_i$. 


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- $e_i^* \neq e_i$. In this case $GL_{n_i}(K_i)$ is nilpotent.
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Assume that $G$ has no 2-elements.

- $G$ is locally finite. Thus we may as well assume that $G$ is finite.
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Let $e_1, \ldots, e_k$ be the primitive central idempotents of $FG$. Then each $FGe_i = M_{n_i}(K_i)$ for some finite field $K_i$.

- $e_i^* \neq e_i$. In this case $GL_{n_i}(K_i)$ is nilpotent and this implies $n_i = 1$.
- $e_i^* = e_i$. 
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Assume that $G$ has no 2-elements.

- $G$ is locally finite. Thus we may as well assume that $G$ is finite.

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Let $e_1, \ldots, e_k$ be the primitive central idempotents of $FG$. Then each $FGe_i = M_{n_i}(K_i)$ for some finite field $K_i$.

- $e_i^* \neq e_i$. In this case $GL_{n_i}(K_i)$ is nilpotent and this implies $n_i = 1$.

- $e_i^* = e_i$. Then there is an induced involution on $M_{n_i}(K_i)$. 
The non-modular case for torsion groups

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- $e_i^* \neq e_i$. In this case $GL_{n_i}(K_i)$ is nilpotent and this implies $n_i = 1$.

- $e_i^* = e_i$. Then there is an induced involution on $M_{n_i}(K_i)$. Now $Un(M_{n_i}(K_i))$ is nilpotent.
The non-modular case for torsion groups

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- $G$ is locally finite. Thus we may as well assume that $G$ is finite.
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Let $e_1, \ldots, e_k$ be the primitive central idempotents of $FG$. Then each $FGe_i = M_{n_i}(K_i)$ for some finite field $K_i$.

- $e_i^* \neq e_i$. In this case $GL_{n_i}(K_i)$ is nilpotent and this implies $n_i = 1$.
- $e_i^* = e_i$. Then there is an induced involution on $M_{n_i}(K_i)$. Now $Un(M_{n_i}(K_i))$ is nilpotent and, as each $ge_i$ is a unitary unit of odd order for all $g \in G$, again $n_i = 1$. 
The non-modular case for torsion groups

Theorem [Lee-Sehgal-S., 2017]

Let $FG$ be the group algebra of a torsion group $G$ over a field $F$ of characteristic $p \neq 2$ endowed with the classical involution, such that $FG$ is non-modular and $G$ has no elements of order 2. Then $Un(FG)$ is bounded Engel and solvable if, and only if, $G$ is abelian.
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Theorem [Lee-Sehgal-S., 2017]

Let $FG$ be the group algebra of a torsion group $G$ over a field $F$ of characteristic $p \neq 2$ endowed with the classical involution, such that $FG$ is non-modular and $G$ has no elements of order 2. Then $Un(FG)$ is bounded Engel and solvable if, and only if, $G$ is abelian.
A crucial example

Example [Lee-Sehgal-S., 2014]

Let $F$ be the field of order 5 and $G$ the dihedral group of order 8. Then $Un(FG)$ is nilpotent, but $U(FG)$ is not bounded Engel and solvable.
The non-modular case

**Theorem [Lee-Sehgal-S., 2017]**

Let $FG$ be the group algebra of a group $G$ over an algebraically closed field $F$ of characteristic $p \neq 2$ endowed with the classical involution, such that $FG$ is non-modular. Then the following are equivalent:

(i) $Un(FG)$ is bounded Engel and solvable;
(ii) $U(FG)$ is nilpotent;
(iii) $G$ is nilpotent and the torsion elements of $G$ are central.


