On some connections between set-theoretic solutions of the Yang-Baxter equation, matrices and noncommutative rings

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Groups, Rings, and the Yang-Baxter equation
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Outline

1. Connections of YBE with geometry, Knot theory, Hopf algebras and other topics

2. Braces, skew braces and the YBE

3. One generator braces

4. Acons and applications in geometry

5. Graded prime rings with Gelfand-Kirillov dimension 2 and differential polynomial rings.
Braces, Yang-Baxter equation geometry and Hopf algebras
A set theoretic solution of the Yang-Baxter equation on
\[ X = \{ x_1, x_2, \ldots, x_n \} \] is a pair \((X, r)\)
where \( r \) is a map \( r : X \times X \to X \times X \) such that:

\[
(r \times I_x)(I_x \times r)(r \times I_x) = (I_x \times r)(r \times I_x)(I_x \times r)
\]

where \( I_x \) is the identity map on \( X \).

Example. If \( r(x_1, x_2) = (x_2, x_1) \) then

\[
(r \times I_x)(x_1, x_2, x_3) = (x_2, x_1, x_3)
\]
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\]

The solution \((X,r)\) is involutive if \( r^2 = id \times X \times X \);

Denote \( r(x; y) = (f(x,y); g(x,y)) \).

The solution \((X,r)\) is nondegenerate if the maps
\( y \to f(x,y) \) and \( y \to g(y,x) \) are bijective, for every \( x \) in \( X \).
Around 2000 non-degenerate set-theoretic solutions have been investigated in a series of fundamental papers by Gateva-Ivanova, Van den Bergh, Etingof, Schedler, Soloviev. In particular the structure group, the permutation group of set-theoretic solutions have been introduced, and the retraction technique for involutive solutions.
Another interesting structure related to the Yang-Baxter equation, the **braided group**, was introduced in 2000, by **Lu, Yan, Zhu**.

In 2015, **Gateva-Ivanova** showed that left braces are in one-to-one correspondence with braided groups with an involutive braiding operator.

Braces and braided groups have **different properties** and can be studied using different methods.
In 2007 Rump introduced braces as a generalization of radical rings related to non-degenerate involutive set-theoretic solutions of the Yang-Baxter equation.

``With regard to the property that A combines two different equations or groups to a new entity, we call A a brace”

Wolfgang Rump

Recently skew-braces have been introduced by Guarnieri and Vendramin to describe all non-degenerate set-theoretic solutions of the Yang-Baxter equation.
Some motivation to study such solutions and connections with geometry, Hopf algebras, Knot theory etc.
A quadratic algebra $A$ is a PBW algebra if there exists an enumeration of $X = \{x_1, \cdots, x_n\}$ such that the quadratic relations form a (noncommutative) Groebner basis with respect to the degree-lexicographic ordering on induced from $x_1 < \cdots < x_n$. 
A class of PBW Arin Schelter regular rings of arbitrarily high global dimension n, were investigated by Gateva-Ivanova, Van den Bergh. It was shown by Gateva-Ivanova and Van den Bergh that they are also closely related to the set-theoretic solutions of the Yang-Baxter equation.
Motivation-geometry

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The problem of classification of Artin Schelter regular PBW algebras with generating relations of type $x_i x_j = q_{i,j} x_{i'} x_{j'}$ and global dimension $n$ is equivalent to the classification of square-free set-theoretic solutions of YBE, $(X, r)$ on sets $X$ of order $n$.  
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T. Gateva-Ivanova
Motivation: Hopf algebras

- There is a connection between non-degenerate, involutive set-theoretic solutions of the YBE with nilpotent rings and braces discovered by Rump in 2007.

- As shown in by Etingof and Gelaki any such solution can be used to construct a minimal triangular Hopf algebra by twisting group algebras.
Motivation: Hopf algebras

• There is a connection between non-degenerate, involutive set-theoretic solutions of the YBE and factorised groups. Many factorised groups can be obtained from nil and nilpotent rings.

• As shown in by Etingof, Gelaki, Guralnick and Saxl any finite factorised group can be used to construct a semisimple Hopf algebra (for example biperfect Hopf algebras)
Motivation: Hopf-Galois extensions

Skew braces correspond to Hopf-Galois extensions (Bachiller, Byott, Vendramin).
Motivation-integrable systems

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Infinite braces and rings may be more important for applications than finite, as finite representations of infinite objects may make it possible to find related spectral parameter dependent solutions of the YBE.
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Robert Weston
Solutions associated with skew braces are \textbf{biquandles}; hence skew braces could be used to construct combinatorial invariants of knots. A \textbf{biquandle} is a non-degenerate set-theoretical solution \((X; r)\) of the YBE such that for each \(x\) in \(X\) there exists a unique \(y\) in \(X\) such that \(r(x; y) = (y; x)\). Biquandles have applications in knot theory.
Braces and skew braces
In 2007 Rump introduced braces as a generalization of radical rings related to non-degenerate involutive set-theoretic solutions of the Yang-Baxter equation.

``With regard to the property that A combines two different equations or groups to a new entity, we call A a brace”

Wolfgang Rump

Recently skew-braces have been introduced by Guarnieri and Vendramin to describe all non-degenerate set-theoretic solutions of the Yang-Baxter equation.
``It is more or less possible to translate all problems of set-theoretic solutions to braces’’ …

``The origin of braces comes to Rump, and he realised that this generalisation of Jacobson radical rings is useful for set-theoretic solutions.’’

David Bachiller
(Algebra seminar, UW, 2015)
**Definition.** A left brace is a set $G$ with two operations $+$ and $\circ$ such that

$(G,+)\text{ is an abelian group,}$

$(G, \circ)\text{ is a group and}$

$$a \circ (b+c) + a = a \circ b + a \circ c$$

for all $a, b, c \in G.$

We call $(G,+)$ the additive group and $(G, \circ)$ the multiplicative group of the right brace.
A right brace is defined similarly, replacing condition

\[ a \circ (b+c)+a = a \circ b + a \circ c \]

by

\[ (a+b) \circ c + c = a \circ c + b \circ c. \]

A two-sided brace is a right and left brace.
Nilpotent ring-product of arbitrary n elements is zero (for some n).
NILPOTENT RINGS AND BRACES
(Rump 2007)

Let N with operations + and · be a nilpotent ring.

The circle operation ◦ on N is defined by

\[ a \circ b = a \cdot b + a + b \]

Two-sided braces are exactly Jacobson radical rings with operations + and ◦.

Intuition: \((a+1)\cdot(b+1)=(a\cdot b+a+b)+1\)
FINITE NILPOTENT RINGS ARE TWO-SIDED BRACES (Rump 2007)
Let \((N, +, )\) be a nilpotent ring. Then \((N, +, \circ )\) is a brace:

* \((N, +)\) is an abelian group

* \[a \circ (-a+aa-aaa+aaaa- \ldots )=0\]

and \[a \circ 0 =a \circ 0 =a\]

Therefore \((N, \circ )\) is a group with the identity element 0.

* \[a \circ (b+c)+a = a(b+c)+a+b+c+a = a \circ b + a \circ c\]
There are many more connections of noncommutative rings with non-degenerate set-theoretic solutions of the YBE via skew braces, as we observed with Leandro Vendramin in our new paper, we will give some examples of such connections.
First we recall definition of skew brace given by Guarnieri and Vendramin.

**Definition.** A skew left brace is a set $G$ with two operations $+$ and $\circ$ such that

$(G,+)$ is a group,

$(G, \circ)$ is a group and

$$a \circ (b+c) + a = a \circ b + (-a)+ a \circ c$$

for all $a, b, c \in G$.

We call $(G,+)$ the additive group and $(G, \circ)$ the multiplicative group of the right brace.
CONNECTIONS WITH THE YANG-BAXTER EQUATION

DID YOU REALLY THINK THE BRACES WOULD HELP?
Let $R$ be a nilpotent ring; then the solution $(R; r)$ of the YBE associated to ring $R$ is defined in the following way: for $x; y \in R$ define

$$r(x; y) = (u; v),$$

where

$$u = x \cdot y + y, \quad v = z \cdot x + x$$

and

$$z = -u + u^2 - u^3 + u^4 - u^5 + ...$$

If $R$ is a left brace $r(x,y)$ is defined similarly:

$$u = x \circ y - x \quad \text{and} \quad v = z \circ x - z$$

where $z \circ u = 0$.

This solution is always non-degenerate and involutive.
It is known (from Rump) that every non-degenerate involutive set-theoretic solution of the Yang-Baxter equation is a subset of a solution associated to some brace $B$,

and hence is a subset of some brace $B$.

**Remark:** A finite solution is a subset of some finite brace (Cedo, Gateva-Ivanova, A.S, 2016).
Theorem (Guarnieri, Vendramin 2016)

Let $R$ be a skew brace then the solution $(R; r)$ of the Yang-Baxter equation associated to ring $R$ is defined in the following way: for $x, y \in R$ define

$$r(x; y) = (u; v),$$

where

$$u = (-x) + x \circ y,$$

and

$$x \circ y = u \circ v.$$ 

This solution is always non-degenerate.
It was shown by Guareni and Vendramin that a large class of non-degenerate involutive set-theoretic solutions, which are called injective solutions, of the Yang-Baxter equation is a subset of a solution associated to some skew left brace $B$.
and hence is a subset of some skew left brace $B$.

Example.
Let $A$ be a group. Then $a \circ b = ab$ is a skew brace.
Similarly,
the operation $a \circ b = ba$ turns $A$ into a skew brace.
Some methods to construct skew braces
Some methods of constructing skew braces

1. Exactly factorised groups, factorised groups related to nil rings, braces and skew braces.

2. Near rings, rings of functions over nil rings.

3. Triply factorised groups, nilpotent rings give many examples of triply factorised groups.
Recall that a group $A$ factorizes through two subgroups $B$ and $C$ if

$$G = AB = \{ab : a \in A; b \in B\}.$$ 

The factorization is said to be exact if $A \cap B = 1$. 

Skew braces and factorized groups
Theorem (Vendramin, A.S, 2017)

Let $A$ be a classical brace (or more generally, a skew brace with nilpotent additive group). Assume that the group $(A; +)$ decomposes as $A_1 + \ldots + A_k$

where the $A_i$ are the **Sylow subgroups** of $(A; +)$. Let $I \subset \{1, 2, \ldots, k\}$.

$$B = \sum_{i \in I} A_i \quad \text{and} \quad C = \sum_{i \notin I} A_i$$

Then $(A; \circ)$ admits an exact factorization $A = B \circ C$
Theorem (Vendramin, 2017)

Let $A$ be a group that admits an exact factorization through two subgroups $B$ and $C$. Then $A$ with

$$a \circ a' = ba'c$$

and

$$a + a' = aa'$$

where $a = bc \in BC$; $a' \in A$;

is a skew brace.
Theorem (Vendramin, A.S., 2017)

Let $R$ be a ring (associative, noncommutative), let $S$ be a subring of $R$ and let $I$ be a left ideal in $R$ such that $S \cap I = 0$ and $R = S + I$. Assume that $S$ and $I$ are Jacobson radical rings (for example nilpotent rings). Then $R$ with the operation

$$a \circ b = a + b + ab$$

is a group and $R = S \circ I$ is an exact factorization.

Therefore, we can construct a skew brace!
Near rings

Sysak has studied the connection between Near rings and triply factorised groups. Hubert introduced construction subgroups in Near rings. Sysak observed that triply factorised groups are related to classical braces.

Recall that near-ring is a set $N$ with two binary operations $+$ and $*$ such that $(N; +)$ is a (not necessarily abelian) group, $(N; *)$ is a semigroup, and

$$x^* (y+z) = x^*y + x^*z$$

for all $x; y; z$ in $N$. 
A subgroup $M$ of the additive group $(N; +)$ of a near ring $N$ is said to be a construction subgroup if $1+M$ is a group.

Theorem (Vendramin, A.S, 2017)
Let $N$ with operations $+$ and $*$ be a near-ring and $M$ be a construction subgroup. Then $M$ is a skew brace with

$$m+m' = m + m'$$
$$m \circ m = m + (1 + m) * m'$$
Theorem (Vendramin, A.S., 2017)
Let $F$ be a finite field and let $A$ be a commutative $F$-algebra such that $A = F + N$ where $N$ is a nilpotent subalgebra of $A$. Let $S$ be the set of all functions from $A$ to $A$ which can be written as polynomials from $N[x]$. Then $S$ with the usual addition and
\[ f(x) \circ g(x) = f(x) + g(x + f(x)) \]
is a classical brace.
Also works for noncommutative $A$ with noncommutative polynomial rings.
This braces has many interesting one-generator subbraces....this is related to ....
A finite non-degenerate solution \((X; r)\) is indecomposable if and only if \(X\) cannot be presented as union of two non-empty sets \(Y, Z\) such that \(r(Y; Y) = (Y; Y)\) and 
\[ r(Z, Z) = (Z, Z). \]
Proposition (A. Smoktunowicz, A.S. 2017)
Let $A$ be a finite left brace and let $x$ in $A$, and let $A(x)$ be the subbrace of $A$ generated by $x$ then

$$X = \{ x+rx : a \in A(x) \},$$

with the usual map $r$, is an indecomposable solution of the YBE
**Question 1.** Characterise one-generator braces of the multipermutation level 2.

**Question 2.** Is the previous proposition also true for skew braces?
Theorem (A. Smoktunowicz, A.S. 2017)

Let \((X; r)\) be a finite solution. The following are equivalent:
1. \((X; r)\) is an indecomposable solution of a finite multipermutation level, and \(x\) is an element of \(X\).
2. There is a finite left brace \(A\) generated by \(x\) such that
   \[X = \{x + ax : a \text{ in } A\}\]
   and \(r\) is as usual. Moreover, \(A\) is a right nilpotent brace.
Theorem (A. Smoktunowicz, A.S. 2017)

Moreover, if \((X; r)\) is a retraction of a solution whose structure group is nilpotent then \(A\) is a left nilpotent brace.
More on examples inspired by Near rings

Theorem (Vendramin, A.S., 2017)

Let $F$ be a finite field and let $A$ be a commutative $F$-algebra such that $A = F + N$ where $N$ is a nilpotent subalgebra of $A$. Let $\circ$ be the circle operation on $N$, so $a \circ b = ab + a + b$. Let $S$ be the set of all functions from $A$ to $A$ which can be written as polynomials from $N[x]$. Then $S$ with the addition

$$f(x) + g(x) := f(x) \circ g(x)$$

and multiplication

$$f(x) \odot g(x) = f(x) \circ g(x \circ f(x));$$

is a skew brace.

Also works for noncommutative $A$ with noncommutative polynomial rings.
Theorem (Vendramin, A.S., 2017)

Why such strange examples-in Near rings many examples comes from the Near ring of functions.

Let G be a (not necessarily abelian) additive group and M(G) be the set of maps $G \rightarrow G$. Then $M(G)$ is a near-ring under the following operations:

$$(f + g)(x) = f(x) + g(x); \quad (f \circ g)(x) = g(f(x));$$

for $f, g$ in $M(G)$
The notions of retract of a solution and multipermutation solution were introduced by Etingof, Schedler and Soloviev.

Rump has shown that a solution associated to a left brace $A$ is a multipermutation solution if and only if

$$A^{(i)} = 0$$

for some $i$, where

$$A^{(1)} = A, \quad A^{(i+1)} = A^{(i)} \cdot A$$
Remark (A.S.)

Any solution contained in a solution of a finite multipermutation level has finite multipermutation level.
MULTIPERMITTED SOLUTIONS

Theorem (A.S. 2015)

If $A$ is a brace whose cardinality is a cube-free number, then $A^{(i)} = 0$ for some $i$.

Moreover, every solution of the YBE contained in $A$ has finite multipermutation level.
BRACES WHICH ARE NOT NILPOTENT

**Theorem** (David Bachiller, 2015).

There exists a finite brace $B$ such that

$$B \cdot B = B.$$ 

Moreover, $B$ has no nontrivial ideals, hence it is a simple brace.
Moreover, $B$ can have 48 elements.
BRACES WHICH ARE NOT NILPOTENT

Many interesting examples of braces which are simple braces, not retractable braces, not nilpotent braces were constructed by Bachiller, Cedo, Jespers, Okninski, Catino, Rizzo, Vendramin and others.
LEFT NILPOTENT AND RIGHT NILPOTENT BRACES

In 2005 Rump introduced radical chains $A^i$ and $A^{(i)}$ where $A^{(1)} = A$

$$A^{(i+1)} = A^{(i)} \cdot A, \quad A^{i+1} = A \cdot A^i$$

If $A^{(i)} = 0$ for some $i$, then we say that $A$ is a right nilpotent brace. If $A^i = 0$ for some $i$, then $A$ is a left nilpotent brace.

Rump have shown that there are finite left nilpotent braces which are not right nilpotent, and finite right nilpotent braces which are not left nilpotent.
LEFT NILPOTENT BRACES

Recall that $A^{i+1} = A \cdot A^i$. If $A^i$ for some $i$, then $A$ is a left nilpotent brace.
Rump have shown that if $A$ is a brace of cardinality $p^i$ for some $i$ and some prime number $p$ then $A$ is a left nilpotent brace.

**Theorem** (A.S. 2015)
Let $A$ be a finite left brace. Then the multiplicative group of $A$ is nilpotent if and only if $A^i = 0$ for some $i$.
Moreover, such a brace is the direct sum of braces whose cardinalities are powers of prime numbers.
RELATED RESULTS FOR RINGS

THEOREM (Amberg, Dickenschied, Sysak 1998)

The following assertions for the following Jacobson radical ring $R$ are equivalent

1. $R$ is nilpotent.

2. The adjoint group of $R$ is nilpotent.
Engel Lie algebras

Theorems

1a. Every n-Engel Lie algebra over a field $K$ of characteristic zero is nilpotent.

1b. An n-Engel Lie algebra over an arbitrary field is locally nilpotent.

2. Any torsion free n-Engel Lie ring is nilpotent.

E. I. Zelmanov.
In 2015 Angiono, Galindo and Vendramin provided Lie-theoretical analogs of braces, and introduced Hopf-Braces.
Thank you very much!
Motivation

Over the last decade there has been a series of new ideas of how to describe properties of certain structures in geometry using noncommutative rings such as reconstruction algebras, MMAs and Acons.

These rings can be described via generators and relations, and they can be studied using Gold-Shafarevich theorem and other methods coming from noncommutative ring theory.
Potential algebras and their versions appear in many different and related contexts in physics and mathematics and are known also under the names vacualgebra, Jacobi algebra, etc.

Let $K\langle x, y \rangle$ be the free associative algebra in two variables, and $F \in K\langle x, y \rangle$ be a cyclically invariant polynomial. We assume that $F$ starts in degree $\geq 3$. 
We consider the potential algebra $A(F)$, given by two relations, which are partial derivatives of $F$, i.e. $A(F)$ is the factor of $K\langle x, y \rangle$ by the ideal $I(F)$ generated by $\partial F/\partial x$ and $\partial F/\partial y$, where for a monomial $w$:

$\partial w/\partial x = u$ if $w = xu$ and 0 otherwise,

$\partial w/\partial y = u$ if $w = yu$ and 0 otherwise.
Example

Let $F=xyy+xyx+yxx$ be our superpotential. Then

$\partial F/\partial x = xy+yx$ and

$\partial F/\partial y = xx$.

Then the potential algebra $A(F) = K<x,y>/I$.

Where $I$ is the ideal generated by $xy+yx$ and $xx$. 
To understand the birational geometry of algebraic varieties via the minimal model program, it is necessary to understand the geometry of certain codimension two modifications known as flips and flops… A central problem is to classify flips and flops in a satisfying manner, and to construct appropriate invariants.

Donovan, Wemyss
We associate a new invariant to every flipping or flopping curve in a 3-dimensional algebraic variety, using noncommutative deformation theory. This generalises and unifies the classical invariants into one new object, the noncommutative deformation algebra $A_{\text{con}}$ associated to the curve. It recovers classical invariants in natural ways. Moreover, unlike these classical invariants, $A_{\text{con}}$ is an algebra.

Donovan, Wemyss
Acons are potential algebras

Acons are certain factors of MMAs- the maximal modification algebras (MMAs); they were developed by Iyama and Wemyss.

If R is a 3-dimensional algebraic variety with MMA $\mathbf{A}$, then by a result of Van den Bergh it follows that the relations of $\mathbf{A}$ come from a superpotential (under mild assumptions). Since Acon is a factor of $\mathbf{A}$ by idempotents, it too comes from a superpotential.
Questions of Wemyss

The potential algebras that come from geometry are finitely dimensional. Wemyss asked several questions

**Question 1.** What is the minimal dimension of an Acon?

**Question 2.** What is the minimal dimension of a potential algebra?

**Question 3.** Do all finitely dimensional potential algebras come from geometry as Acons?

Wemyss et al. proved that rings coming from geometry have special central elements and are of a special form.
Some new results on Acons

Theorem (N. Iyudu, A.S.)
Let $A(F)$ be a potential algebra given by a potential $F$ having only terms of degree 5 or higher. Then the potential algebra $A(F)$ is infinite dimensional and has exponential growth. Moreover, growth of a potential algebra whose potential $F$ has only terms of degree 4 or higher can be polynomial.

Question. If $F$ has terms of degree 4 or higher, can the potential algebra $A(F)$ be finite-dimensional?
Minimal degree of an Acon

Theorem (N.Iyudu, A.S.)
The dimension of every potential algebra is at least 8. Therefore, the dimension of every Acon is at least 8.

M. Wemyss showed that the potential algebra (Acon) with $F=\text{xxxy+xyx+yxx+xxx+yyyy}$ has degree 9. Idea of the proof: Consider algebra $A(F)/A(F)^{(5)}$ and then apply Diamond Lemma to this algebra in the ‘reverse gradation’ where elements of smaller degree have bigger gradation than elements of smaller degrees. It works because ring is nilpotent.
Some results on prime rings with Gelfand-Kirillov dimension 2
Nil algebra-every element to some power is zero.

Nilpotent algebra-product of arbitrary n elements is zero (for some n).
Graded-nil ring is a graded ring in which every homogeneous element is nilpotent.
Baer radical, also called the prime radical is the intersection of all prime ideals in the ring. It is always locally nilpotent.
The Jacobson radical

The ring $R/J(R)$ has zero Jacobson radical, so the Jacobson radical is useful for removing ‘bad elements’ from a ring.

Nathan Jacobson
Differential polynomial rings over Baer radical are locally nilpotent

**Theorem** (Greenfeld, Ziembowski, A.S. 2017)

Let $R$ be ring with derivation $D$, and $R[x; D]$ be the differential polynomial ring.

If $R$ is Baer radical then $R[x; D]$ is locally nilpotent.

**Corollary** (Greenfeld, Ziembowski, A.S. 2017)

Let $R$ be an algebra over characteristic 0 and $D$ be a derivation on $R$. If $R$ has a nilpotent ideal then $R[x; D]$ has a locally nilpotent ideal.
On Z-graded algebras

Theorem (Greenfeld, Leroy, Ziembowski, A.S. Alg. Repres. Th.) Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be an affine, prime, $\mathbb{Z}$-graded algebra over a field $K$. Suppose that $R_0$ is finite-dimensional, and that $R$ is generated in degrees $1$, $-1$ and $0$. Suppose that $R_k \neq 0$, for almost all $k$. Then $R$ has no nonzero graded-nil ideals. In particular, the Jacobson radical of $R$ is zero, so $R$ is semiprimitive. Moreover, $R_0$ is semiprimitive.
A K-algebra $A$ has GK dimension $d$ if $A$ has approximately $n^d$ elements of degree less than $n$ linearly independent over $K$. Algebras with finite GK dimension have polynomial growth.
Gelfand-Kirillov dimension

Gelfand-Kirillov of a finitely generated commutative algebra is equal to its Krull dimension

A. Kirillov
Classical Krull dimension

Recall that an algebra $R$ over a field $K$ has a classical Krull dimension equal to $m$ if there exists a chain of prime ideals

$$P_m \not\subseteq \ldots \not\subseteq P_1 \not\subseteq P_0 = R$$

of length $m$ and there is no such chain longer than $m$. 
Theorem (Greenfeld, Leroy, Ziembowski, A.S. Alg. Repres. Th.) Let $R$ be an affine, prime algebra over a field $K$ with quadratic growth, which is $\mathbb{Z}$-graded and generated in degrees 1, $-1$ and 0. We write $R = \bigoplus_{i \in \mathbb{Z}} R_i$, and assume that $R_0$ is finite-dimensional. Then $R$ has finite classical Krull dimension.
Thank you very much!