On the Zassenhaus conjecture for direct products

Mariano Serrano

with Andreas Bächle and Wolfgang Kimmerle

Department of Mathematics
University of Murcia

Groups, Rings and the Yang-Baxter equation
Spa, June 22, 2017
1 The Zassenhaus conjecture
  • An introduction
    • Direct products and Camina groups
    • The Extended HeLP Method
Integral group rings

$G$ a finite group. $\mathbb{Z}G$ the integral group ring of $G$.

$$U(\mathbb{Z}G) = \{\text{Units of } \mathbb{Z}G\}$$

The elements $\pm g$ with $g \in G$ are called trivial units.

The Augmentation map

\[ \varepsilon : \mathbb{Z}[G] \to \mathbb{Z} \]
\[ \sum_{g \in G} u_g g \to \sum_{g \in G} u_g \]

Units with augmentation one
- \( V(\mathbb{Z}[G]) = \{ u \in U(\mathbb{Z}[G]) : \varepsilon(u) = 1 \} \).
- \( U(\mathbb{Z}[G]) = \pm V(\mathbb{Z}[G]) \).

General problem
How are the torsion elements of \( V(\mathbb{Z}[G]) \)?
The Zassenhaus Conjecture (1974)

Every torsion unit of $V(\mathbb{Z}G)$ is conjugate in $U(\mathbb{Q}G)$ to an element of $G$.

The Zassenhaus Conjecture has been proved for:

- Nilpotent groups. (Weiss 1991)
- Groups having a normal Sylow subgroup with abelian complement. (Hertweck 2006)
- Cyclic-by-abelian groups. (Caicedo, Margolis and del Río 2013)
- $\text{PSL}(2, p)$ for $p$ a Fermat or Mersenne prime. (Margolis, del Río and S. 2016)
- Groups till order 143. (Bächle, Herman, Konovalov, Margolis and Singh 2016)
$x^G$ the conjugacy class of $x$ in $G$.

Partial augmentations in $\mathbb{Z}G$

- $u = \sum_{g \in G} u g g \in \mathbb{Z}G$ and $x \in G$.
- $\varepsilon_x(u) = \sum_{g \in x^G} u g$ the **partial augmentation** of $u$ at $x$. 
Mariano Serrano

### On the Zassenhaus conjecture for direct products

Marciniak, Ritter, Sehgal and Weiss.

A finite group. The following conditions are equivalent:

1. The Zassenhaus Conjecture holds for $G$.
2. For every torsion element $u \in V(\mathbb{Z}G)$, every $d \mid |u|$ and every $x \in G$ we have $\varepsilon_x(u^d) \geq 0$. 

A result to deal with the Zassenhaus conjecture
The Zassenhaus conjecture
An introduction
Direct products and Camina groups
The Extended HeLP Method

1 The Zassenhaus conjecture
  • An introduction
  • Direct products and Camina groups
  • The Extended HeLP Method
General problem for direct products

Open problem

$G$ and $H$ finite groups satisfying the Zassenhaus conjecture. Does the Zassenhaus conjecture hold for $G \times H$?
Höfert 2004

$G$ finite group for which the Zassenhaus conjecture holds. Then it also holds for $G \times C_2$.

Hertweck 2008

$G$ finite group for which the Zassenhaus conjecture holds. $H$ nilpotent group with $\gcd(|G|, |H|) = 1$. Then the Zassenhaus conjecture holds for $G \times H$.

Goal

$G$ finite group, $A$ abelian finite group. Study the Zassenhaus conjecture for $G \times A$. 
Camina groups

$G'$ the derived subgroup of the finite group $G$.

**Definitions (1978)**

- $G$ is called a **Camina group** if $gG' = g^G$ for every $g \in G \setminus G'$.
- For a positive integer $n$, a Camina group $G$ is called an **$n$-Camina group** if $G'$ is the union of $n$ $G$-conjugacy classes.

**Examples**

- 1-Camina groups are precisely the abelian finite groups.
- $S_3$, $A_4$ and $D_8$ are 2-Camina groups.
- $C_2^4 \rtimes C_3$ is a 6-Camina group.
The general classification by Dark and Scoppola 1996

A finite non-abelian group is a Camina group if and only if it is a Camina $p$-group or a Frobenius group whose complement is either cyclic or $Q_8$.

Remark

The Zassenhaus conjecture holds for Camina groups.
Theorem (Bächle, Kimmerle and S. 2017)

$G$ Camina group. A abelian finite group. Then the Zassenhaus conjecture holds for $G \times A$. 
Another approach

\( \zeta_m \) a complex primitive \( m \)-th root of unity.

**Hertweck 2008**

\( G \) finite group. A abelian finite group with exponent \( m \).
Suppose that any torsion unit in \( V(\mathbb{Z}[\zeta_m]G) \) is conjugate in \( U(\mathbb{Q}(\zeta_m)G) \)
to an element of \( G \).
Then the Zassenhaus conjecture holds for \( G \times A \).
The Extended HeLP Method

$G$ finite group.
$\mathcal{O}$ the ring of algebraic integers in a number field $K$.

<table>
<thead>
<tr>
<th>HeLP Method</th>
<th>Extended HeLP Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u \in V(\mathbb{Z}G)$ of order $n$</td>
<td>$u \in V(\mathcal{O}G)$ of order $n$</td>
</tr>
<tr>
<td>$\varepsilon_x(u) \in \mathbb{Z}$ for any $x \in G$</td>
<td>$\varepsilon_x(u) \in \mathbb{Z}[\zeta_n] \cap \mathcal{O}$ for any $x \in G$</td>
</tr>
</tbody>
</table>

**Idea of the method**

- Produce restrictions over $\varepsilon_x(u^d)$ for any $x \in G$ and any $d \mid n$.
- (MRSW’87) $u$ is conjugate in $U(KG)$ to an element of $G$ iff $\forall d \mid n$, all but one of the partial augmentations of $u^d$ vanish.
The result

Work in progress (Bächle, Kimmerle, S. 2017)

$G$ a finite group with $|G| \leq 95$. A abelian finite group.
$G$ doesn’t map onto $S_4$, $G \not\cong A_5$ and $G \not\cong (S_3 \times S_3) \rtimes C_2$.
Then the Zassenhaus conjecture holds for $G \times A$. 
Theorem (Bächle, Kimmerle and S. 2017)

Let \( G = (P_1 \rtimes A_1) \times \cdots \times (P_k \rtimes A_k) \) be a finite group where \( P_i \) are \( p_i \)-groups, \( A_i \) are abelian groups and \( P_1 \times \cdots \times P_k \) is a Hall subgroup of \( G \). Then the Zassenhaus conjecture holds for \( G \).

Corollary

\( H \) a finite group with a normal nilpotent Hall subgroup \( N \) such that \( H/N \) is abelian. Then \( H \) can be embedded into a group \( G \) for which the Zassenhaus conjecture holds.
Thanks for your attention.  

Officially not celebrating Eric Jespers’ 62nd birthday