Algebras of Linear Growth and the Dynamical Mordell–Lang Conjecture

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Graded algebras

We call an associative algebra $A$ graded if

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots,$$

where $A_0 = k$ is a basic field, $\dim_k A_i < \infty$. All our algebras are graded.

Hilbert function: $h_A(n) = \dim A_n$

Hilbert series:

$$H_A(z) = \sum_{n \geq 0} z^n \dim A_n = \sum_{n \geq 0} z^n h_A(n).$$
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Rationality

An algebra $A$ is called **finitely presented** if it is defined by a finite number of generators and relations.

**Theorem (Govorov, 1972)**

If the relations of a finitely presented algebra $A$ are monomials in generators then $H_A(z)$ is a rational function.

**Conjecture (Govorov)**

For each finitely presented algebra $A$ the Hilbert series $H_A(z)$ is a rational function.


Open questions: Govorov conjecture for Noetherian algebras and for Koszul algebras.
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Rationality in the linear growth case

An algebra has **linear growth**, if $\text{GK-dim } A \leq 1$, that is, for some $c > 0$ we have $h_A(n) = \dim A_n < c$.

**Example**

Let $A = \langle x, y| x^2, yxy, xy^{2^t}x \text{ for all } t \geq 0 \rangle$. Then $A_n = k\{y^n, xy^{n-1}, y^{n-1}x, xy^{n-2}x\}$ for $n \neq 2^t + 2$ or $A_n = k\{y^n, xy^{n-1}, y^{n-1}x\}$ otherwise.

We have $H_A(z) = 1 + 2z + 4z^2/(1 - z) - z^2 \sum_{t \geq 0} z^{2^t}$.

**Problem (Govorov conjecture for algebras of linear growth, GALG)**

*Suppose that an algebra $A$ of linear growth is finitely presented. Is $H_A(z)$ a rational function?*

For such algebras, $H_A(z)$ is rational iff $h_A(n)$ is eventually periodic, that is, $\exists n_0, T > 0$ such that $h_A(n) = h_A(n + T)$ for all $n > n_0$. 
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Automaton algebras

Let $X$ be a finite generating set of an algebra $A$. Consider a multiplicative ordering `$<$' of the set of all words in $X$. A word on $X$ is called normal in $A$ if it is not a linear combination of less words. The set $N$ of all normal words is a linear basis of $A$.

**Definition (Ufnarovski)**

An algebra $A$ is called automaton if $N$ is a regular language.

Recall that a language is regular iff it is recognized by a finite automaton.
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Property (generalized Govorov theorem): If $A$ is graded automaton (that is, it is graded and automaton with homogeneous $X$ and a degree-compatible ordering ‘$<$’), then $H_A(z)$ is a rational function.

Conjecture (Ufnarovski, 1990)
A finitely presented algebra of linear growth is automaton.

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UGA implies GALG.
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The finite characteristic case

**Theorem**

Suppose that the field $k$ has a finite characteristic. Then both Govorov conjecture for algebra of linear growth and Ufnarovski conjecture for graded algebras hold if and only if $k$ is an algebraic extension of its prime subfield.

‘If’ part: essentially, the case of finite field.

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The case of infinite field

What about the case char $k = 0$?

Example (Fermat algebras)

For $\alpha, \beta \in k^\times$, let $A = A_{\alpha, \beta}$ be generated by $a, b, c, x, y, z$ subject to 26 relations $xc - \alpha cx, yb - \beta cy$ and others. Then $h_A(n + 3)$ is 10 or 11 according to whether the Fermat equality $\alpha^n + \beta^n = 1$ holds. So, it has no nonzero solution in $k^\times$ for each $n \geq 3$ if and only if $h_A(i) = 10$ for all $i \geq 6$ and each $A = A_{\alpha, \beta}$.

Theorem

Let $g \geq 5$ be an integer. If the field $k$ is infinite, then there are infinitely many (periodic) sequences $h_A$ for $g$-generated quadratic $k$-algebras of linear growth. If in addition, $k$ contains all primitive roots of unity, then both the length $d$ of the initial non-periodic segment and the period $T$ of $h_A$ can be arbitrary large.
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Skolem–Mahler–Lech theorem

**Theorem (Skolem–Mahler–Lech)**

*If char \( k = 0 \) and \( a_n = c_1a_{n-1} + \cdots + c_d a_{n-d} \) is a linear recurrence over \( k \), then the zero set \( \{n \geq 0 | a_n = 0\} \) is the finite union of several arithmetic progressions and a finite set.*

We refer to a set of nonnegative integers which is the union of a finite set and a finite collection of arithmetic progressions as **SML**. All known proofs essentially use \( p \)-adic analysis.

**Example (Lech, 1953)**

If the field \( k \) has characteristic \( p > 0 \) and \( t \in k \) is transcendental over the prime subfield \( F_p \) then the sequence \( a_n = (t + 1)^n - t^n - 1 \) satisfies \( a_n = 0 \) iff \( n = p^m \) with \( m \geq 0 \).
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The dynamical Mordell–Lang conjecture: formulation

A challenging generalization:

**Conjecture (The dynamical Mordell–Lang conjecture)**

Let $\mathcal{V}$ be a quasiprojective variety over $k$ (of characteristic zero), let $\Phi : \mathcal{V} \to \mathcal{V}$ be any morphism, and let $\alpha \in \mathcal{V}$. Then for each subvariety $Y \subset \mathcal{V}$, the set $\{n \geq 0 | \Phi^n(\alpha) \in Y\}$ is SML.

See the book (which is unfortunately absent in our library...) : Jason P. Bell, Dragos Ghioca, Thomas J. Tucker, *The Dynamical Mordell-lang Conjecture*, AMS, 2016 (Mathematical Surveys and Monographs, vol. 210)
The dynamical Mordell–Lang conjecture: some results

Some important cases are known.

**Theorem (Bell, 2006)**

Let $\text{char } k = 0$. The conjecture is true provided that $\mathcal{V}$ is an affine variety and $\Phi$ is a polynomial automorphism.

Note that this corollary implies the Skolem–Mahler–Lech theorem.

**Theorem**

For any field $k$, GALG implies the linear dynamical Mordell–Lang conjecture.

So, we have implications

$\text{UGA} \rightarrow \text{GALG} \rightarrow \text{Linear ML} \rightarrow \text{Skolem–Mahler–Lech}$. 
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Generalized dynamical Mordell–Lang conjecture

Does dML implies UGA or (at least) GALG?
Let $A$ is generated and related in degrees at most $D$. Let $V = \text{mod-} A^{\leq D}$ be the category of graded modules with generators and relations in degrees $0..D$. Then $F : M \mapsto M_{\geq 1}[1]$ is an endofunctor of $V$. For $m > 0$, let $V_m = \{ M \in \text{Ob } V | \dim M_0 = m \}$.

Then GALG is equivalent to the following categorical version of dML: Given a finitely presented graded algebra $A$ of linear growth as above and a positive integer $m$, is the set $\{ n \geq 0 | F^n(A) \in V_m \}$ always SML?
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Proof of the main theorem

‘Only if’ $k$ an algebraic extension of $F_p \implies$ UGA $\implies$ GALG

For example, let us show: $k$ is finite $\implies$ GALG

*Proof.* There are finite number (say, $N$) of isomorphism classes of $M \in \text{mod-}A_{\leq D}$ such that $\dim M_0 \leq C$ (where $\dim A_n \leq C$ for all $n \geq 0$). Then the sequence $h_A(n) = \dim F^n(A)_0$ is periodic with both period and non-periodic segment of length at most $N$.

‘If’

$k$ contains $F_p[t] \implies$ GALG fails.

Using Lech example ($a_n = (t + 1)^n - t^n - 1$): there is a 6-generated algebra with generators $a, b, c, x, y, z$ with relations $xc - (t + 1)cx, yc - tcy, zc - cz$ and 23 others (the same as for Fermat algebras). Then

$$h_A(n + 3) = \begin{cases} 11, & n = p^m \text{ for some } m \geq 0, \\ 10, & \text{otherwise.} \end{cases}$$
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‘If’ \( k \) contains \( F_p[t] \implies \) GALG fails.

Using Lech example (\( a_n = (t+1)^n - t^n - 1 \)): there is a 6-generated algebra with generators \( a, b, c, x, y, z \) with relations \( xc - (t+1)cx, yc - tcy, zc - cz \) and 23 others (the same as for Fermat algebras). Then

\[
h_A(n + 3) = \begin{cases} 
11, & n = p^m \text{ for some } m \ge 0, \\
10, & \text{otherwise.}
\end{cases}
\]
Proof of the main theorem

‘Only if’ $k$ an algebraic extension of $F_p \implies$ UGA $\implies$ GALG

For example, let us show: $k$ is finite $\implies$ GALG

Proof. There are finite number (say, $N$) of isomorphism classes of $M \in \text{mod-} A^{\leq D}$ such that $\dim M_0 \leq C$ (where $\dim A_n \leq C$ for all $n \geq 0$). Then the sequence $h_A(n) = \dim F^m(A)_0$ is periodic with both period and non-periodic segment of length at most $N$.

‘If’

$k$ contains $F_p[t] \implies$ GALG fails.

Using Lech example ($a_n = (t + 1)^n - t^n - 1$): there is a 6-generated algebra with generators $a, b, c, x, y, z$ with relations $xc - (t + 1)cx, yc - tcy, zc - cz$ and 23 others (the same as for Fermat algebras). Then

$$h_A(n + 3) = \begin{cases} 
11, & n = p^m \text{ for some } m \geq 0, \\
10, & \text{otherwise.}
\end{cases}$$
Normal words in algebras of linear growth

The next Proposition follows from [Belov and others, 1997].

\begin{quote}
\textbf{Proposition}

If a (non-graded) algebra $A$ of linear growth is generated by a finite set $S$, then there are $U, V, W \subseteq S^*$ such that each normal word in $A$ has the form

$$w = ac^n b, \text{ where } a \in U, b \in V, c \in W, n \geq 0.$$ 

\end{quote}

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\textbf{Proposition}

Suppose a language $L$ consists of the words of the above form. Then $L$ is regular iff for each $a \in U, b \in V, c \in W$ the set

$$N_{a,b,c} = \{ n \in \mathbb{Z}_+ | ac^n b \in L \}$$

is SML.

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Normal words in algebras of linear growth

Corollary

Suppose that the algebra $A$ is automaton. Then there are a generating set $1 \in S \subset A$ and an ordering such that for some $Q \subset S^3$ the set of normal words in $A$ is

$$\{ac^nb|n \geq 0, (a, b, c) \in Q\}.$$
Thank you very much!