Quantized Coordinate Rings and Universal Bialgebras

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Definition

Let $q \in \mathbb{K}^\times$ and $\mathcal{O}_q(M_N) := \mathbb{C}\langle t_{ij} | 1 \leq i, j \leq N \rangle/(\text{Rel})$ where $\text{Rel}$:

\[
\begin{align*}
& t_{ik}t_{il} - qt_{il}t_{ik} \\
& t_{ik}t_{jk} - qt_{jk}t_{ik} \\
& t_{jk}t_{il} - t_{il}t_{jk} \\
& t_{ik}t_{jl} - t_{jl}t_{ik} - (q - q^{-1})t_{il}t_{jk}
\end{align*}
\]

$(\forall i < j, \ k < l)$

$\exists \hat{R} \in \mathbb{C}^{N^2 \times N^2}$ such that $\hat{R}$ satisfies the Yang-Baxter Eq. and

$$(\text{Rel}) = \left( \sum_{k,l} \hat{R}_{kl}^{ij}t_{km}t_{ln} - t_{ik}t_{jl} \hat{R}_{mn}^{kl} | i,j,m,n \leq N \right)$$
Quantum Matrices

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\]
Properties of $\mathcal{O}_q(M_N)$

Assumption: $q \in \mathbb{k}^\times$ is not a root of unity.

$\mathcal{O}_q(M_N)$ is a bialgebra in a natural way, s.t.

- it is cosemisimple,
- dimensions of its simple comodules are the same as of $\mathcal{O}(M_N)$,
- (Domokos, Lenagan, 2003) the representation ring of its comodules, that is $\mathcal{O}_q(M_N)^{\text{coc}}$, is a commutative polynomial ring of rank $N$,
- (Sz. M., 2015) $\mathcal{O}_q(M_N)^{\text{coc}}$ is a maximal commutative subalgebra.
Properties of $O_q(M_N)$

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Ring theory of $\mathcal{O}_q(M_N)$

$\mathcal{O}_q(M_N)$ is a fin. pres. graded algebra generated in degree 1, s.t.

1. it is a Noetherian domain,
2. noncommutative UFD, catenary, etc.,
3. it is a PBW-algebra, i.e. ordered monomials in $t_{11}, t_{12}, \ldots, t_{nn}$ form a basis in $\mathcal{O}_q(M_N)$,
4. its relations are of the form $(\hat{R}TT - TT\hat{R})$ for some $\hat{R}$ s.t.

   - Yang-Baxter Eq.: $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$
   - Hecke equation: $(\hat{R} + 1)(\hat{R} - q) = 0$

$\mathcal{T}(V)/\text{Im}(\hat{R} + 1)$ and $\mathcal{T}(V)/\text{Im}(\hat{R} - q)$ are PBW-algebras.

Theorem (Sudbery)

2) $\Rightarrow$ 1)
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$\mathcal{P}(V)/\text{Im}(\hat{R} + 1)$ and $\mathcal{P}(V)/\text{Im}(\hat{R} - q)$ are PBW-algebras.

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Theorem (Sudbery)

2) $\Rightarrow$ 1)
Universal bialgebras

Definition/Proposition (Manin, Takeuchi, Sudbery, ...)

For any decomp. \( V^\otimes 2 = S \oplus T \),

\[
\mathcal{M}(S, T) := \mathcal{T}(\text{End}(V))/\tau_{23}(S \otimes S^\perp + T \otimes T^\perp)
\]

is a bialgebra coacting on \( \mathcal{T}(V)/(S) \) and \( \mathcal{T}(V)/(T) \).

For \((S, T)\):

- \(\text{Sym}^2(V), \Lambda^2(V) \hookrightarrow \mathcal{O}(M_N)\)
- \(\text{Sym}^2_q(V), \Lambda^2_q(V) \hookrightarrow \mathcal{O}_q(M_N)\) where
  \[
  \Lambda^2_q(V) = \langle x_i x_j - qx_j x_i \mid 1 \leq i < j \leq N \rangle \\
  \text{Sym}^2_q(V) = \langle qx_i x_j + x_j x_i \mid 1 \leq i < j \leq N \rangle
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For $(S, T)$:

- $\text{Sym}^2(V), \Lambda^2(V) \hookrightarrow \mathcal{O}(M_N)$
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**Definition/Proposition (Manin, Takeuchi, Sudbery, ...)**

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Basic properties of $\mathcal{M}(S, T)$

$\mathcal{M} := \mathcal{M}(S, T)$ is a

- finitely generated, quadratic algebra,
- $\dim \mathcal{M}_1 = N^2$ and $\dim \mathcal{M}_2 = (\dim S)^2 + (\dim T)^2$,

Assume that

$$\dim S = \dim \text{Sym}^2(V) = \binom{N+1}{2}$$

hence $\dim T = \dim \Lambda^2(V) = \binom{N}{2}$ and

$$\dim \mathcal{M}_2 = \dim \mathcal{O}(\mathcal{M}_N)_2 = \binom{N^2+1}{2}.$$

Question

When does $\mathcal{M}$ have a PBW-basis, i.e. an ordered basis in $\mathcal{M}_1$ s.t. ordered monomials form a basis in $\mathcal{M}$? In particular, $\dim \mathcal{M}_3 =$?
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**Question**

When does $\mathcal{M}$ have a PBW-basis, i.e. an ordered basis in $\mathcal{M}_1$ s.t. ordered monomials form a basis in $\mathcal{M}$? In particular, $\dim \mathcal{M}_3 =$?
Existence of an $R$-matrix

Theorem (Sz. M.)

Let $(V, \langle ., . \rangle)$ be a fin. dim. Hilbert space, $S \subseteq V \otimes V$, $T = S^\perp$. If

$$\dim(S \otimes V \cap V \otimes S) = \binom{N+2}{3} \quad \dim(S \otimes V \cap V \otimes T) = 0$$

then

$$\dim \mathcal{M}_3 \leq \dim \mathcal{O}(M_3) = \binom{N^2+2}{3}$$

with equality if and only if $\text{Rel}(\mathcal{M}) = (\hat{R}TT - TT\hat{R})$ for some $\hat{R} \in \text{End}(V \otimes V)$ s.t.

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}$$

Moreover, in this case, $\hat{R}$ satisfies the Hecke equation.
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Moreover, in this case, \(\hat{R}\) satisfies the Hecke equation.
Four subspace quiver

Observation (generalizing Raedschelders, Van den Bergh)

As an algebra

$$\mathcal{M}_n^\vee \cong$$
Four subspace quiver

Observation (basically by M. Van den Bergh)
As an algebra

\[ M_n^\vee \cong \{ e \in \text{End}(V^{\otimes n}) \mid e(U) \subseteq U, \]
\[ U \in \{ S_{i,i+1}, T_{i,i+1} \mid i = 1, \ldots, k-1 \} \}

where \( S_{i,i+1} = V^{\otimes (i-1)} \otimes S \otimes V^{\otimes (n-i-1)} \) and similarly for \( T \).

Hence,

\[ M_3^\vee \cong \text{End}_{\tilde{D}_4}(\rho) \]

for the rep. \( \rho = (V^{\otimes 3}, S_{12}, S_{23}, T_{12}, T_{23}) \) of the (tame Euclidean) quiver \( \tilde{D}_4 \).
Four subspace quiver

Observation (basically by M. Van den Bergh)

As an algebra

$$\mathcal{M}_n^\vee \cong \{ e \in \text{End}(V^\otimes n) \mid e(U) \subseteq U,\ 
\text{such that} \ U \in \{S_{i,i+1}, T_{i,i+1} \mid i = 1, \ldots, k-1\}\}$$

where $S_{i,i+1} = V^{\otimes (i-1)} \otimes S \otimes V^{\otimes (n-i-1)}$ and similarly for $T$.

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$$\mathcal{M}_3^\vee \cong \text{End}_{\tilde{D}_4}(\rho)$$

for the rep. $\rho = (V^\otimes 3, S_{12}, S_{23}, T_{12}, T_{23})$ of the (tame Euclidean) quiver $\tilde{D}_4$. 
Orthogonal case

Special case: $V$ is a fin. dim. Hilbert space, $T = S^\perp$.

- The config. $(S_{12}, S_{23}, S_{12}^\perp, S_{23}^\perp)$ is determined by

$$\text{Spec} \left( \text{Proj}_{S_{12}} \circ \text{Proj}_{S_{23}} \circ \text{Proj}_{S_{12}} \right)$$

as a multiset (Name in real case: principal angles.).

- Fewer distinct angles $\Rightarrow$ more endomorphisms of $\rho$.

- $S_{12} \cap S_{23}$ gives eigenvalue 1, $S_{12}^\perp$ gives 0,

- If $\lambda, \mu \in \text{Spec}(P_{S_{12}} P_{S_{23}} P_{S_{12}}) \backslash \{0, 1\}$, $\lambda \neq \mu$ then $\nexists \hat{R}$ for $\mathcal{M}$.

- Greatest dim $\mathcal{M}_3 \iff S_{12}$ and $S_{23}$ are isoclinic modulo $S_{12} \cap S_{23} \iff \nexists \hat{R}$
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Orthogonal case

Special case: $V$ is a fin. dim. Hilbert space, $T = S^\perp$.

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- Application: Takeuchi’s conjecture: quantum orthogonal bialgebra $\tilde{\mathcal{M}}_+(3) \rightarrow O_q(\mathfrak{so}_3)$ has a PBW-basis. It is false!

- Notes:
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  - further directions: related Hopf algebras, corepresentation theory of $\tilde{\mathcal{M}}$, further ring theoretical properties of $\tilde{\mathcal{M}}$. 
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Thank you for your attention!
References

- T. Hayashi, Quantum deformations of classical groups, Publ. RIMS Kyoto Univ. 28 (1992), 57-81.