Structure groups of YBE solutions: cohomological applications

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\[(ab)c = a(bc)\]

\[
z^{-1}(y^{-1}xy)z = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)
\]
Yang–Baxter equation

Data:

- monoidal category $\mathcal{C} (= \text{Vect}_k)$;
- object $S$;
- morphism $r: S \otimes S \to S \otimes S$.

YBE: $r_1r_2r_1 = r_2r_1r_2: S^{\otimes 3} \to S^{\otimes 3}$

$r_1 = r \otimes \text{id}_S$, $r_2 = \text{id}_S \otimes r$
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Braiding YBE:
$$r_1 r_2 r_1 = r_2 r_1 r_2: S^\otimes 3 \to S^\otimes 3$$

$r_1 = r \otimes \text{Id}_S$, $r_2 = \text{Id}_S \otimes r$

Topological avatar:

YBE $\iff$ Reidemeister III move
We’ll mostly work with set-theoretic solutions: $C = \textbf{Set}$ (Drinfel’d ’90).

linearise $\longrightarrow$ deform $\longrightarrow$ linear solutions.
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linearise $\quad$ deform $\quad$ linear solutions.

Example: $r(x, y) = (y, x)$

$\quad$ R-matrices;
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- linearise
- deform

\( \leadsto \) \( \leadsto \) linear solutions.

**Example:** \( r(x, y) = (y, x) \)

\( \leadsto \) R-matrices;

\( \leadsto \) \( r_{\text{Lie}} (x \otimes y) = y \otimes x + \mathfrak{h} \mathfrak{l} \otimes [x, y] \):

\[ \text{YBE for } r_{\text{Lie}} \iff 1 \text{ central Jacobi for } [\ ] \]
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linearise   deform  $\Rightarrow$ linear solutions.

Example: $r(x, y) = (y, x)$

$\Rightarrow$ R-matrices;

$\Rightarrow$ $r_{\text{Lie}}(x \otimes y) = y \otimes x + h1 \otimes [x, y]$:

YBE for $r_{\text{Lie}} \iff$ Jacobi for $[]$

1 central

Example: $r_{SD}(x, y) = (y, x \triangleright y)$:

YBE for $r_{SD} \iff$ self-distributivity for $\triangleright$

Self-distributivity: $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$
Self-distributivity

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YBE for \( r_{SD} \) ⇔ self-distributivity for \( \triangleleft \)

Self-distributivity: \( (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z) \)

Examples:

- group \( S \) with \( x \triangleleft y = y^{-1}xy \):
- abelian group \( S \), \( t: S \rightarrow S \), \( a \triangleleft b = ta + (1 - t)b \).
3 Self-distributivity

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• group $S$ with $x \triangleleft y = y^{-1}xy$:
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Applications:
• invariants of knots and knotted surfaces (Joyce & Matveev ’82);

\[
\begin{pmatrix}
  y & x \triangleleft y \\
  \downarrow & \downarrow \\
  x & y
\end{pmatrix}
\]
Self-distributivity

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\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
\text{y} & \text{x} & \text{y} \\
\text{x} & \text{y} & \text{y}
\end{array}
\]

- Hopf algebra classification \((Andruskiewitsch–Graña '03).\)
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Example: **Involutive solutions** \( r \), i.e., \( r^2 = \text{Id}_{S \times S} \).

A solution \( r(a, b) = (\sigma_a(b), \tau_b(a)) \) is called **left non-degenerate (LND)** if the maps \( \tau_b \) are bijective.

**Theorem** (*Rump ’04*): LND involutive solutions \( \iff \) cycle sets.
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**Theorem** (Rump ’04): LND involutive solutions $\leftrightarrow$ cycle sets.

**Theorem** (Soloviev & Lu–Yan–Zhu ’00, L.–Vendramin ’17):

- LND solution $(S, r) \sim$ SD operation $\triangleleft_r$ on $S$;
- $\triangleleft_r$ captures major properties of $r$; for instance, $r^2 = \text{Id}_{S \times S} \iff a \triangleleft_r b = a$. 
### Two axes

**Example:** Involution solutions $r$, i.e., $r^2 = \text{Id}_{S \times S}$.

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**Theorem (Soloviev & Lu–Yan–Zhu ’00, L.–Vendramin ’17):**

- LND solution $(S, r) \mapsto$ SD operation $\prec_r$ on $S$;
- $\prec_r$ captures major properties of $r$; for instance,

$$r^2 = \text{Id}_{S \times S} \iff a \prec_r b = a.$$

So, involutive and self-distributive solutions can be seen as two perpendicular axes in the space of all LND solutions. Schematically,

"$0 \rightarrow \text{CycleSets} \rightarrow \text{LNDSol} \rightarrow \text{SD} \rightarrow 0$"
Getting more exotic

We will tolerate non-invertible solutions.

Example: free self-distributive structures.

Application: total order on braid groups (Dehornoy ’91).
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  Application: total order on braid groups (*Dehornoy* ’91).

Even worse: some of our solutions are idempotent: $rr = r$.

**Examples**: ✓ Monoid $(S, \cdot, 1)$, $r_{\text{Ass}}(x, y) = (1, x \cdot y)$:

YBE for $r_{\text{Ass}} \iff$ 1 unit associativity for $\cdot$
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  YBE for \( r_{\text{Ass}} \)  ⇐⇒ 1 unit associativity for \( \cdot \)

- ✓ Factorised monoid \( G = HK \), \( S = H \cup K \), \( r_{\text{Fact}}(x, y) = ((xy)_H, (xy)_K) \).
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![YBE for \( r_{\text{Ass}} \) ⇔ 1 unit associativity for \( \cdot \)]

 ✓ Factorised monoid \( G = HK \), 

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S = H \cup K, \quad r_{\text{Fact}}(x, y) = ((xy)_H, (xy)_K).
\]

 ✓ Lattice \((S, \wedge, \lor)\), \( r_{\text{L}}(x, y) = (x \wedge y, x \lor y) \).
Abstract nonsense?

So, YBE provides a unifying framework for many algebraic situations.

Question: Can anything non-trivial be done in such a general setting?
Abstract nonsense?

So, YBE provides a unifying framework for many algebraic situations.

**Question:** Can anything non-trivial be done in such a general setting?

**Answer:** Yes!

1) A study of structure groups of solutions.

2) A (co)homology theory.
Why should a group theorist care about YBE?

Structure group, or universal enveloping group of \((S, r)\):

\[
G(S, r) = \langle S \mid xy = y'x' \text{ whenever } r(x, y) = (y', x') \rangle
\]

Structure monoids and algebras are defined similarly.
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G(S, r, e) := G(S, r)/e = 1
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Examples:

\checkmark Factorised monoid \(G = HK\), \(r_{\text{Fact}}(x, y) = ((xy)_H, (xy)_K)\):

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\text{Mon}(H \cup K, r_{\text{Fact}}, 1_G) \simeq G.
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✓ Factorised monoid \(G = HK, \ r_{\text{Fact}}(x, y) = ((xy)_H, (xy)_K)\):
  \[
  \text{Mon}(H \cup K, r_{\text{Fact}}, 1_G) \cong G.
  \]

✓ Lie algebra \(V, \ V' = V \oplus k1, 1 \text{ central}, \ r_{\text{Lie}}(x \otimes y) = y \otimes x + 1 \otimes [x, y]\):
  \[
  \text{Alg}(V', r_{\text{Lie}}, 1) \cong UEA(V, [\ ]).
  \]
Why should a group theorist care about YBE?

\[ G(S, r) = \langle S \mid xy = y'x' \text{ whenever } r(x, y) = (y', x') \rangle \]

Strategy (Cedó–Jespers–del Río ’10):
Step 1: classify all structure groups \( G \) (or certain quotients thereof);
Step 2: classify all YBE solutions with \( G(S, r) \cong G \).
Why should a group theorist care about YBE?

\[ G(S, r) = \langle S \mid xy = y'x' \text{ whenever } r(x, y) = (y', x') \rangle \]

YBE solutions \rightarrow \text{groups & algebras} \rightarrow \text{methods}

\begin{align*}
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\end{align*}

Theorem: \( r^2 = \text{Id} \implies \)

\begin{itemize}
    \item [✓] \( \text{Mon}(S, r) \) is of I-type, cancellative, Ore;
    \item [✓] \( \text{Grp}(S, r) \) is solvable, Garside, Bieberbach;
    \item [✓] \( \mathbb{k} \text{Mon}(S, r) \) is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular
\end{itemize}

(Manin, Gateva-Ivanova & Van den Bergh, Etingof–Schedler–Soloviev, Jespers–Okniński, Chouraqui 80’–…).
Braided cohomology

Construction (Fenn et al. ’93, Carter et al. ’04, L. ’13):

✓ $C^n := \text{Maps}(S \times^n, \mathbb{Z}_m)$;
✓ $d^n : C^n \to C^{n+1}$, 
\[
d^n = \sum_{i=1}^{n+1} (-1)^{i-1} (d_{l; i}^{n,i} - d_{r; i}^{n,i}).
\]

Versions:

✓ diagrammatic:

\[
f(x'_1 \cdots x'_{i-1} x_{i+1} \cdots x_{n+1})
\]
\[
\uparrow
\]
\[
x'_i x'_1 \cdots x'_{i-1} x_{i+1} \cdots x_{n+1}
\]
\[
r_1 \cdots r_{i-1} \uparrow
\]
\[
x_1 \cdots x_{n+1}
\]
\[
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\]
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f
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Braided cohomology

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$$r_1 \ldots r_{i-1} \uparrow$$

$$x_1 \ldots x_{n+1}$$

✓ a topological realisation;
✓ using quantum shuffles;
✓ using a differential graded bialgebra (Farinati–García-Galofre ’16).
Why I like braided cohomology

1. Describes diagonal deformations (Freyd–Yetter ’89, Eisermann ’05):

   \[ r_\omega(x, y) = q^{\omega(x, y)} r(x, y), \quad \omega: S \times S \to \mathbb{Z}_m. \]

   \[ d^2 \omega = 0 \implies r_\omega \text{ is a YBE solution.} \]
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2. Yields knot and knotted surface invariants (Carter et al. ’01):
   \((S, r)\)-coloured diagram \((D, C)\) & \(\omega : S \times S \to \mathbb{Z}_m\)
   \(\rightsquigarrow \) Boltzmann weight \(B_\omega(C) = \sum_{y'} \omega(x, y') - \sum_{x'} \omega(x, y')\).
Why I like braided cohomology

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   \((S, r)\)-coloured diagram \((D, C)\) \& \(\omega: S \times S \to \mathbb{Z}_m\)

   \[ \leadsto \text{Boltzmann weight } \mathcal{B}_\omega(C) = \sum \omega(x, y) - \sum \omega(x, y) \]  

   \[ d^2 \omega = 0 \implies \sum C t^{\mathcal{B}_\omega(C)} \text{ is a knot invariant}; \]

   \[ \omega - \omega' = d^1 \psi \implies \omega \text{ and } \omega' \text{ yield equivalent invariants}. \]
Why I like braided cohomology

Unifies cohomology theories for

✓ self-distributive structures
  \( r_{SD}(x, y) = (y, x \triangleleft y) \)

✓ associative structures
  \( r_{Ass}(x, y) = (1, x \cdot y) \)

✓ Lie algebras
  \( r_{Lie}(x \otimes y) = y \otimes x + h1 \otimes [x, y] \)

+ explains parallels between them,
+ suggests theories for new structures:

Example: cycle sets and braces (L.–Vendramin ’17).
Why I like braided cohomology

3. Unifies cohomology theories for
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4. Computes the cohomology of structure groups.
Comparing cohomologies

Quantum symmetriser $QS$:

<table>
<thead>
<tr>
<th>Braided cohomology</th>
<th>$H^* (S, \mathbb{Z}_m)$</th>
<th>$\cup$</th>
<th>Small complexes</th>
</tr>
</thead>
</table>

| Hochschild cohomology | $HH^* (\text{Mon}(S, r), \mathbb{Z}_m)$ | $\cup$ | Tools |

$\xrightarrow{QS}$
Comparing cohomologies

Quantum symmetriser $QS$:

- braided cohomology $H^*(S, \mathbb{Z}_m)$
- cup product $\smile$
- small complexes

$QS$ \hspace{1cm} \text{Hochschild cohomology} $HH^*(\text{Mon}(S, r), \mathbb{Z}_m)$

\hspace{1cm} cup product $\smile$

\hspace{1cm} tools

Theorem: $QS$ is an isomorphism when

- $rr = \text{Id}$ (Farinati–García-Galofre ’16);
- $rr = r$ (L. ’17).

Open question: For general $r$?
Comparing cohomologies

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- Hochschild cohomology $HH^*(\text{Mon}(S, r), \mathbb{Z}_m)$

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Applications:

- Spectral sequence for factorised monoids $G = HK$. 
Comparing cohomologies

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- Braided cohomology: $H^* (S, \mathbb{Z}_m) \xleftarrow{QS} HH^* (\text{Mon}(S, r), \mathbb{Z}_m)$
- Cup product $\cup$
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Open question: For general $r$?

Applications:
- Spectral sequence for factorised monoids $G = HK$.
- Cohomology computations for plactic monoids.