Almost Engel finite, profinite, and compact groups

Evgeny Khukhro

University of Lincoln, UK,
and Sobolev Institute of Mathematics, Novosibirsk

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Joint work with Pavel Shumyatsky

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Engel groups

Notation: left-normed simple commutators

\[ [a_1, a_2, a_3, \ldots, a_r] = [...[[a_1, a_2], a_3], \ldots, a_r]. \]

Recall: a group $G$ is an **Engel group** if for every $x, g \in G$,

\[ [x, g, g, \ldots, g] = 1, \]

where $g$ is repeated sufficiently many times depending on $x$ and $g$.

Clearly, any locally nilpotent group is an Engel group.
Known facts on finite groups

**Zorn’s Theorem**

*A finite Engel group is nilpotent.*

**Proof:**

Coprime action $\Rightarrow$ non-Engel.

No coprime action $\Rightarrow$ nilpotent.

**Baer’s Theorem**

*If g is an Engel element of a finite group G, that is, $[x, g, \ldots, g] = 1$ for every $x \in G$, then $g \in F(G)$.*

Here, $F(G)$ is the Fitting subgroup, largest normal nilpotent subgroup.
Engel compact groups

J. Wilson and E. Zelmanov, 1992

*Any Engel profinite group is locally nilpotent.*

Proof relies on

**Zel’manov’s Theorem**

*If a Lie algebra $L$ satisfies a nontrivial identity and is generated by $d$ elements such that each commutator in these generators is ad-nilpotent, then $L$ is nilpotent.*

Yu. Medvedev, 2003

*Any Engel compact (Hausdorff) group is locally nilpotent.*
Almost Engel groups

Definition

A group $G$ is almost Engel if for every $g \in G$ there is a finite set $E(g)$ such that for every $x \in G$,

$$[x, g, g, \ldots, g]_n \in E(g) \quad \text{for all } n \geq n(x, g).$$

Includes Engel groups: when $E(g) = \{1\}$ for all $g \in G$.

Theorem 1 (Almost Engel $\Rightarrow$ almost locally nilpotent)

Suppose that $G$ is an almost Engel compact (Hausdorff) group. Then $G$ has a finite normal subgroup $N$ such that $G/N$ is locally nilpotent.

(...And there is also a locally nilpotent subgroup of finite index: $C_G(N)$.)
Three parts of the proof

1. **Finite groups**, a quantitative version.

2. **Profinite groups**: using finite groups, Wilson–Zelmanov theorem.

3. **Compact groups**: reduction to profinite case using structure theorems for compact groups.
Some notation

If $G$ is an almost Engel group, then for every $g \in G$ there is a unique minimal finite set $\mathcal{E}(g)$ with the property that for every $x \in G$,

$$[[x, g, g, \ldots, g]] \in \mathcal{E}(g) \quad \text{for all } n \geq n(x, g)$$

(for possibly larger numbers $n(x, g)$).

We fix the symbols $\mathcal{E}(g)$ for these minimal sets, call them Engel sinks.

The nilpotent residual of a group $G$ is

$$\gamma_\infty(G) = \bigcap_i \gamma_i(G),$$

where $\gamma_i(G)$ are terms of the lower central series ($\gamma_1(G) = G$, and $\gamma_{i+1}(G) = [\gamma_i(G), G]$).
Almost Engel finite groups

For finite groups there must be a quantitative analogue of the hypothesis that the sinks $\mathcal{E}(g)$ are finite.

**Theorem 2**

Suppose that $G$ is a finite group and there is a positive integer $m$ such that $|\mathcal{E}(g)| \leq m$ for every $g \in G$. Then $|\gamma_\infty(G)|$ is bounded in terms of $m$.

(...And $G$ also has a nilpotent normal subgroup of bounded index: $C_G(\gamma_\infty(G))$.)

Theorem 2 can be viewed as a generalization of Zorn’s theorem that a finite Engel group is nilpotent: almost Engel $\Rightarrow$ almost nilpotent.
About the proof for finite groups

Lemma
In any almost Engel group $G$, a (minimal) Engel sink is the set

$$\mathcal{E}(g) = \{ z \in G \mid z = [z, g, \ldots, g] \}$$

(with at least one occurrence of $g$).

Indeed, $x \rightarrow [x, g]$ is a mapping of $\mathcal{E}(g)$ into itself, must be “onto” since $\mathcal{E}(g)$ is finite and minimal, $z \in \mathcal{E}(g)$ belongs to its orbit.

Lemma
In a finite group, if $A$ is an abelian section, acted on by $g$ of coprime order, then $[A, g] = \{ [a, g, \ldots, g] \mid a \in A \}$ for any number of $g$, so $[A, g] \subseteq \mathcal{E}(g)$.

Proof: $C_{[A, g]}(g) = 1 \quad \Rightarrow \quad [A, g] = \{ [b, g] \mid b \in [A, g] \}$. 

□
About the proof for finite groups

Lemma

Let $V$ be an elementary abelian $q$-group, and $U$ a $q'$-group of automorphisms of $V$. If $|[V, u]| \leq m$ for every $u \in U$, then $|[V, U]|$ is $m$-bounded, and therefore $|U|$ is also $m$-bounded.

Lemma

If $|\mathcal{E}(g)| \leq m$ for all $g \in G$, then $G/F(G)$ is of $m$-bounded exponent.

Proof: Clearly, $g$ centralizes its powers. Hence for any $z \in \mathcal{E}(g^k)$ we have

$$z = [z, g^k, \ldots, g^k] \implies z^g = [z^g, g^k, \ldots, g^k].$$

Therefore $\mathcal{E}(g^k)$ is $g$-invariant. Choose $k = m!$. Then $g^{m!}$ centralizes $\mathcal{E}(g^{m!})$, hence $\mathcal{E}(g^{m!}) = \{1\}$ in fact, so $g^{m!}$ is an Engel element. By Baer’s theorem, then $g^{m!} \in F(G)$, so $G/F(G)$ has exponent dividing $m!$. 

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Further proof for finite groups

Proposition

If $\forall |E(g)| \leq m$, then $|G/F(G)|$ is $m$-bounded.

First for the case of soluble $G$.

Then considering the generalized Fitting subgroup = socle of $G/S(G)$ (using CFSG).

Finally Theorem 1 (that $|\gamma_\infty(G)|$ is $m$-bounded)

is proved by induction on $|G/F(G)|$...
Profinite groups

Inverse limits of finite groups.

Topological groups. Quotients only by closed subgroups.

Open subgroups have finite index and are also closed.

Sylow theory. Pronilpotent (=pro-(finite nilpotent)) groups are Cartesian products of pro-$p$ groups.

Largest normal pronilpotent subgroup (closed).
Lemma

A pronilpotent almost Engel group $H$ is in fact an Engel group.

Proof: For any $h \in H$ there is a normal subgroup $R$ such that $E(h) \cap R = \{1\}$ with nilpotent $H/R$.

Then $E(h) \subseteq R$, so in fact $E(h) = \{1\}$,

so $h$ is an Engel element.
Bounded version for profinite groups

Theorem 2 on finite groups immediately implies the following.

**Corollary**

Suppose that $G$ is an almost Engel profinite group and there is a positive integer $m$ such that $|E(g)| \leq m$ for every $g \in G$. Then $G$ has a finite normal subgroup $N$ of order bounded in terms of $m$ such that $G/N$ is locally nilpotent.

**Proof:** In each finite quotient, $\gamma_\infty$ has $m$-bounded order by Theorem 2.

Then $N = \gamma_\infty(G)$ has $m$-bounded order.

$G/N$ is pronilpotent; by Lemma above is an Engel group.

Then by Wilson–Zelmanov theorem $G/N$ is locally nilpotent.
General case of profinite groups

Theorem 3

Suppose that $G$ is an almost Engel profinite group. Then $G$ has a finite normal subgroup $N$ such that $G/N$ is locally nilpotent.

Cannot simply apply Theorem 2 on finite groups – as there is no apriori uniform bound on $|E(g)|$.

First goal: a pronilpotent normal subgroup of finite index.

In the proof, a certain section is considered, and the Baire category theorem is applied.
A piece of proof

Lemma

In an almost Engel profinite group $G$, the sets

$$E_k = \{x \mid |\mathcal{E}(x)| \leq k\}$$

are closed.

Proof: For $y \notin E_k$ we have $|\mathcal{E}(y)| \geq k + 1$, so there are $z_1, z_2, \ldots, z_{k+1}$ distinct elements, each

$$z_i = [z_i, y, \ldots, y].$$

There is an open normal subgroup $N$ such that the images of the $z_i$ are distinct in the finite quotient $G/N$.

Then equations (*) show that for every $n \in N$ the sink $\mathcal{E}(yn)$ has an element in every coset $z_iN$, whence $|\mathcal{E}(yn)| \geq k + 1$. So $yN$ is also contained in $G \setminus E_k$. Thus, $G \setminus E_k$ is open, so $E_k$ is closed.

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Application of Baire theorem

Recall: $E_k = \{ x \mid |E(x)| \leq k \}$ are closed.

In the theorem, $G$ is almost Engel, which means $G = \bigcup E_k$.

By the Baire category theorem, one of $E_k$ contains an open set, coset $aU$, where $U$ is an open subgroup.

This gives us, in a certain metabelian section, a uniform bound for $|E(u)|$ for all $u \in U$, and then Theorem 2 on finite groups can be applied...

Thus, $|G/F(G)|$ is finite, where $F(G)$ is the largest pronilpotent normal subgroup (which is also locally nilpotent by Lemma above). Further arguments are by induction on $|G/F(G)|$ and are similar to those for finite groups.
Compact groups

Recall

**Theorem 1**

*Suppose that $G$ is an almost Engel compact group. Then $G$ has a finite normal subgroup $N$ such that $G/N$ is locally nilpotent.*

Structure theorems for compact groups:

- The connected component $G_0$ of the identity is a divisible group (that is, for every $g \in G_0$ and every integer $k$ there is $h \in G_0$ such that $h^k = g$).
- $G_0/Z(G_0)$ is a Cartesian product of simple compact Lie groups.
- $G/G_0$ is a profinite group.

Note that a simple compact Lie group is a linear group.
Lemma

An almost Engel divisible group is an Engel group.

Proof: For $g \in G_0$, let $|\mathcal{E}(g)| = m$. Choose $h \in G_0$ such that $h^{m!} = g$. Clearly, $h$ centralizes $g$, so for any $z \in \mathcal{E}(g)$ we have

$$z = [z, g, \ldots, g] \implies z^h = [z^h, g, \ldots, g].$$

Hence $\mathcal{E}(g)$ is $h$-invariant. Then $h^{m!} = g$ centralizes $\mathcal{E}(g)$. This means that actually $\mathcal{E}(g) = \{1\}$, so $g$ is an Engel element.

By the structure theorem, $G_0$ is divisible, so is Engel by the above. By well-known results (Garashchuk–Suprunenko, 1960) linear Engel groups are locally nilpotent. Hence $Z(G_0) = G_0$ is abelian by the structure theorem.
Using the profinite case

We apply Theorem 3 on profinite groups to $G/G_0$.

Thus we have $G_0 < F < G$ with $G_0$ abelian divisible, $F/G_0$ finite, and $G/F$ locally nilpotent.

Next steps:

\[ \mathcal{E}(g) \cap G_0 = \{1\} \text{ for all } g \in G; \]

\[ [G_0, \mathcal{E}(g)] = 1 \text{ for all } g \in G; \]

Replace (rename) $F$ by possibly smaller subgroup $\langle \mathcal{E}(g) \mid g \in G \rangle G_0$, so $G_0 \leq Z(F)$;

... etc., in the end use Theorem 3 on profinite again.
Rank restriction (work in progress)

Instead of being finite, suppose that $E(g)$ generates a subgroup of finite (Prüfer) rank, for all $g \in G$.

Conjecture:
If $G$ is a compact (or profinite) group, then there is a normal closed subgroup $N$ of finite rank such that $G/N$ is locally nilpotent.

So far, the case of finite groups seems to have been done:

Theorem 4
Suppose that $G$ is a finite group and there is a positive integer $r$ such that $\langle E(g) \rangle$ has rank at most $r$ for every $g \in G$. Then the rank of $\gamma_\infty(G)$ is bounded in terms of $r$. 