Application of the graded Posner theorem

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Definition

Let $G$ be a group and $\mathbb{F}$ be a field. An (associative) $\mathbb{F}$-algebra $A$ is called $G$-graded if

$$A = \bigoplus_{g \in G} A_g,$$

where $A_g$ is an $\mathbb{F}$-subspace of $A$ and $A_g A_h \subseteq A_{gh}$ for every $g, h \in G$. 

A is $G$-simple if it does not have any non-trivial graded ideals.

Example (Fine grading)

$A = \mathbb{F} \alpha H = \bigoplus_{g \in H} \mathbb{F} \cdot u_g$, where $H$ is a subgroup of $G$. 
Group graded algebras

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More examples

Example

Take $G = \mathbb{Z}/2\mathbb{Z} = \{0, \bar{1}\}$ and consider

$$A_0 = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}; \quad A_{\bar{1}} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ w & z & 0 \end{pmatrix}$$

In short, $$\begin{pmatrix} \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} \\ \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$$
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Let $G$ be a group and $\overrightarrow{g} = (g_1, \ldots, g_n) \in G^{\times n}$. The $G$-graded $\mathbb{F}$ algebra $A = M_{\overrightarrow{g}}(\mathbb{F})$ is the algebra $M_n(\mathbb{F})$ graded by:
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$$
\begin{pmatrix}
    g_1^{-1}g_1 & g_1^{-1}g_2 & g_1^{-1}g_3 & \cdots & g_1^{-1}g_n \\
    g_2^{-1}g_1 & g_2^{-1}g_2 & g_2^{-1}g_3 & \cdots & g_2^{-1}g_b \\
    g_3^{-1}g_3 & g_3^{-1}g_3 & \cdots & \cdots & \cdots \\
    \vdots & \vdots & \ddots & \cdots & \cdots \\
    g_n^{-1}g_1 & g_n^{-1}g_2 & \cdots & \cdots & g_n^{-1}g_n = e \\
\end{pmatrix}
$$
Suppose $A$ is a $G$-graded $\mathbb{F}$-algebra such that:

1. $K = \mathbb{Z}(A)$ is a field.
2. $A$ satisfies an ordinary PI.
3. $A$ is $G$-semiprime (this holds if $G$ is finite and $J(A) = 0$).

Then, $A$ is a $G$-simple finite dimensional $K$-algebra. This holds for $\mathbb{F}$ of characteristic zero and $G$ a residually finite group. If one replaces "$G$-semiprime" by "$G$-semisimple", the theorem is true for every $G$. The main ingredient in the proof is the existence of a special kind of central polynomials:
Suppose $A$ is a $G$-graded $\mathbb{F}$-algebra such that:

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- The main ingredient in the proof is the existence of a special kind of central polynomials:
Strong central polynomials

Definition

A polynomial \( f(x_1, ..., x_n) \in F \langle X \rangle \) is a **strong central polynomial of exponent** \( d^2 \) **for the group** \( G \), if:

1. \( f \) is central (non-identity) for every \( G \)-simple algebra \( A \) of exponent \( d^2 \).

2. For every homogeneous elements \( a_1, ..., a_n \) (i.e., they are inside \( \bigcup_{g \in G} A_g \)),
   \[
   f(a_1, ..., a_n) \neq 0 \iff f(a_1, ..., a_n)e \neq 0.
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It turns out that when \( F \) is of characteristic zero, every central polynomial for \( A \) is strong for every group \( G \).
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A theorem of Aljadeff and Haile

Theorem

Let $G$ be a group and $\mathbb{F}$ an algebraically closed field of $0$ characteristic and $A, B$ two f.d. $G$-simple $\mathbb{F}$-algebras.

Notice that the theorem is easy when $G = \{e\}$. Indeed, $A = \mathbb{M}_n(\mathbb{F})$, $B = \mathbb{M}_m(\mathbb{F})$ and $n^2 = \exp(A)$ is determined by $\text{id}(A)$.

We now show how one can use Posner's theorem in order to get a quick proof of this theorem.
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Let $G$ be a group and $\mathbb{F}$ an algebraically closed of 0 characteristic and $A, B$ two f.d. $G$-simple $\mathbb{F}$-algebras. Then, $A$ is $G$-isomorphic to $B$ iff $\text{id}_G(A) = \text{id}_G(B)$.

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Let $G$ be a group and $\mathbb{F}$ an algebraically closed field of characteristic 0 and $A, B$ two f.d. $\mathbb{F}$-algebras. Then, $A$ is $G$-isomorphic to $B$ iff $id_G(A) = id_G(B)$.

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- We now show how one can use Posner's theorem in order to get a quick proof of this theorem.
Let $A$ be a $G$-graded $\mathbb{F}$-algebra. The ($G$-graded) relatively free algebra of $A$ is defined to be

$$U_A := \mathbb{F} \langle X_G \rangle / id_G(A).$$
Generic algebras

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Let $A$ be a $G$-graded $F$-algebra. The ($G$-graded) relatively free algebra of $A$ is defined to be

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If $A$ is also f.d., then it is possible to embed $U_A$ in a form of $A$:

That is, there is a field $\mathbb{L}_A$ for which $U_A$ is embedded inside $A_{\mathbb{L}_A} := A \otimes_F \mathbb{L}_A$. 


We know that \( id_G(A) = id_G(B) \), so \( \mathbb{F}\langle X \rangle / id_G(B) \simeq \mathbb{F}\langle X \rangle / id_G(B) \).
Proof of the theorem

- We know that $id_G(A) = id_G(B)$, so 
  $\mathbb{F}\langle X \rangle / id_G(B) \cong \mathbb{F}\langle X \rangle / id_G(B)$.

- In other words,

\[ U_A \cong U_B \]

\[ A_{L_A} \cong B_{L_B} \]
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U_A \\ \downarrow \\
\simeq \\
\downarrow \\
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\end{array} \quad \begin{array}{c}
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\simeq \\
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\end{array}$$

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- We want to use Posner’s theorem, but the center is not be a field...
- The evaluation of \( Z(U_A)_e \) to \( A \), is inside \( Z(A)_e = \mathbb{F} \cdot 1_A \). So, for \( 0 \neq f \in Z(U_A)_e \) and \( 0 \neq g \in U_A \), we have an evaluation \( f(\overrightarrow{a}) \cdot g(\overrightarrow{a}) = \alpha g(\overrightarrow{a}) \neq 0 \), where \( 0 \neq \alpha \in \mathbb{F} \).
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- We may consider the algebras $S_A := Z(U_A)_e^{-1}U_A \simeq Z(U_B)_e^{-1}U_B =: S_B$. Both are over $\mathbb{K}$ =the quotient field of $Z(U_A)_e = Z(U_B)_e$ and have the same $G$-graded identities as $A$ (and $B$).
Using Posner’s theorem

- Now we can use Posner’s theorem:

\[
Z(SA) = K
\]

which is a field.

Clearly \( A \) is PI (since f.d.), so also \( SA \).

\( J(UA) = \emptyset \), since an evaluation of \( f \in J(UA) \) into \( A \) is inside \( J(A) = \emptyset \). Hence, also \( JS(A) = \emptyset \).

So, \( SA \) and \( SB \) are \( G \)-simple and f.d. over \( K \).
Now we can use Posner’s theorem:

- We made $Z(S_A) = K$ which is a field.
Using Posner’s theorem

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  - We made $Z(S_A)_e = \mathbb{K}$ which is a field.
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- Now we can use Posner’s theorem:
  - We made $Z(S_A)_e = K$ which is a field.
  - Clearly $A$ is PI (since f.d.), so also $S_A$.
  - $J(U_A) = 0$, since an evaluation of $f \in J(U_A)$ into $A$ is inside $J(A) = 0$. Hence, also $J(S_A) = 0$. 

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- So, $S_A$ and $S_B$ are $G$-simple and f.d. over $\mathbb{K}$. 
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- So, $S_A$ and $S_B$ are $G$-simple and f.d. over $\mathcal{K}$.
- We have the picture:

\[
\begin{array}{ccc}
U_A & \xrightarrow{\sim} & U_B \\
\downarrow & & \downarrow \\
S_A & \xrightarrow{\sim} & S_B \\
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A_{\mathcal{L}_A} & \sim & B_{\mathcal{L}_B}
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Finishing the proof

What is left to show is that $S_A$ is a form of $A$. Since then it follows that $A$ and $B$ are graded forms of each other.
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$$S_A \otimes_K \mathbb{L}_A \rightarrow S_A \cdot \mathbb{L}_A$$

is a graded isomorphism and

$$S_A \cdot \mathbb{L}_A = A_{\mathbb{L}_A}.$$
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The first part follows from the fact that $S_A \otimes_K \mathbb{L}_A$ is $G$-simple.
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- For this it is enough to show that the natural map

$$S_A \otimes_K L_A \to S_A \cdot L_A$$

is a graded isomorphism and

$$S_A \cdot L_A = A_{L_A}.$$ 

- The first part follows from the fact that $S_A \otimes_K L_A$ is $G$-simple.
- The second part follows from

$$\dim_{L_A} A_{L_A} = \exp_G(A_{L_A}) = \exp_G(S_A \cdot L_A) = \dim_{L_A} S_A L_A.$$
The End!

Thank you!