Strong Lie derived length of group algebras vs. derived length of their group of units

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Groups, Rings and the Yang-Baxter equation
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What we want to do

Let $R$ be an associative ring with unity, and $U := U(R)$ be its group of units, and
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\begin{align*}
\delta_0(U) &= U \\
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\delta_i(U) &= (\delta_{i-1}(U), \delta_{i-1}(U))
\end{align*}
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$U$ is said to be **solvable** if $\delta_n(U) = 1$ for some $n$; the smallest such $n$ is denoted by $dl(U)$ and called the **derived length** of $U$.

For $x, y \in R$ set $[x, y] = xy - yx$.

$R$ is called **strongly Lie solvable** if $\delta^{(n)}(R) = 0$ for some $n$; the smallest such $n$ is denoted by $dl^L(R)$ and called the **strong Lie derived length** of $R$.

If $x, y$ are units, then $(x, y) = 1 + x^{-1}y^{-1}[x, y]$.

If $R$ is strongly Lie solvable, then $U$ is solvable with $dl(U) \leq dl^L(R)$. 

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A third series

Let $\delta^0(R) = R$, and for $i \geq 1$, let $\delta^i(R)$ be the additive subgroup of $R$ generated by all Lie commutators $[x, y]$ with $x, y \in \delta^{i-1}(R)$.

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Strongly Lie solvable group algebras

Write $FG$ for the group algebra (or group ring) of a group $G$ over a field $F$.

Passi-Passman-Sehgal, 1973:
$FG$ is strongly Lie solvable iff either $G$ is abelian, or char $F = p$ and $G'$ is a finite $p$-group.

Classification of group algebras with solvable group of units

Bovdi, 2005:
Let char$(F) = p > 3$ and $G$ be a group with a nontrivial $p$-Sylow subgroup $P$ such that if $G$ is non-torsion, then $P$ is infinite. Then $U(FG)$ is a solvable iff $FG$ is strongly Lie solvable.

In the sequel we suppose that $FG$ is strongly Lie solvable.
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Then $dl_L(FG) \leq dl^L(FG)$ and $dl(U(FG)) \leq dl^L(FG)$.
When the derived length is at most 2

Levin-Rosenberger, 1986:
FG is (strongly) Lie metabelian iff one of the following conditions holds:
- $G$ is abelian;
- $\text{char}(F) = 3$, and $G'$ is central of order 3;
- $\text{char}(F) = 2$, and $G'$ is central elementary abelian of order dividing 4.

Shalev, 1991:
Let $\text{char}(F) > 2$ and $G$ be a group. $U(FG)$ is metabelian iff $FG$ is (strongly) Lie metabelian.

Kurdics, 1996; Coleman-Sandling, 1998:
Let $\text{char}(F) = 2$ and $G$ be a nilpotent group. $U(FG)$ is metabelian iff $FG$ is (strongly) Lie metabelian.
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Baginski, 2002:
If $\text{char}(F) = p > 2$, and $G$ is a finite $p$-group with cyclic commutator subgroup, then

$$\text{dl}(U(FG)) = \lceil \log_2(|G'| + 1) \rceil.$$

Questions:
- What happens if $G$ is not a finite $p$-group?
- What happens for $p = 2$?
- How much is $\text{dl}^L(FG)$ in these cases?
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Theorem (Balogh-J, 2008)

Let $FG$ be a (strongly) Lie solvable group algebra. If $G'$ is cyclic of order $p^n$, where $p > 2$, and $G/C_G(G')$ has order $2^m p^r s$, where $(2p, s) = 1$, then

$$dl_L(FG) = dl^L(FG) = \lceil \log_2 2p^n \nu_m \rceil,$$

where $\nu_m = 1$ if $s > 1$, otherwise $\nu_m = 1 - \frac{1}{2^{m+1}}$.

$G$ is nilpotent $\Rightarrow$ $G/C_G(G')$ is a $p$-group $\Rightarrow$ $m = 0, s = 1$ $\Rightarrow$ $\nu_0 = 1/2$ $\Rightarrow$

$$dl_L(FG) = dl^L(FG) = \lceil \log_2 p^n \rceil = \lceil \log_2 (p^n + 1) \rceil = \lceil \log_2 (|G'| + 1) \rceil$$
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$$dl_L(FG) = dl_L(FG) = \lceil \log_2 p^n \rceil = \lceil \log_2(p^n + 1) \rceil = \lceil \log_2(|G'| + 1) \rceil.$$
The derived length of the group of units

**Corollary**

If \( \text{char}(F) = p > 2 \), and \( G \) is a finite \( p \)-group with cyclic commutator subgroup, then

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dl(U(FG)) = dl_L(FG) = dl^L(FG) = \lceil \log_2(|G'| + 1) \rceil
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Balogh-Li, 2007:

If \( G \) is torsion, or non-nilpotent, and \( G' \) is a cyclic \( p \)-group, then

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**Theorem (J, 2016)**

\( dl(U(FG)) \) does not always equal to \( dl^L(FG) \) for nilpotent and non-torsion \( G \).
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**Theorem (J, 2016)**

$\text{dl}(U(FG))$ does not always equal to $\text{dl}^L(FG)$ for nilpotent and non-torsion $G$. 
When the derived length may be smaller

**Theorem (J, 2016)**

Let $G$ be a nilpotent group whose commutator subgroup is finite abelian, and let $\text{char}(F) = p > 2$. If $G' = \text{Syl}_p(G)$, and $\gamma_3(G) \subseteq (G')^p$, then

$$\text{dl}(U(FG)) \leq \left\lceil \log_2 \left( \frac{2}{3} \left(t(G') + 1\right) \right) \right\rceil.$$ 

Furthermore, if $G'$ is cyclic, then the equality holds.

**Theorem (J, 2006)**

Let $G$ be a nilpotent group with $G'$ a finite $p$-group, and let $\text{char}(F) = p$. If $\gamma_3(G) \subseteq (G')^p$, then $\text{dl}^L(FG) = \lceil \log_2(t(G') + 1) \rceil$. 

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Examples

Example

Let

\[ G = \langle a, b, c \mid c^5 = 1, b^{-1}ab = ac, ac = ca, bc = cb \rangle, \]

and \( \text{char}(F) = 5 \). Then \( \text{dl}(U(FG)) = 2 \), but \( \text{dl}_L(FG) = \text{dl}^L(FG) = 3 \).

Example

Let \( G' = \text{Syl}_3(G) \cong C_3 \times C_3 \), \( G' \) is central, and let \( \text{char}(F) = 3 \). Then \( \text{dl}(U(FG)) = 2 \), but \( \text{dl}_L(FG) = \text{dl}^L(FG) = 3 \).

Corollary

\( U(FG) \) can be metabelian, even if \( FG \) is strongly Lie solvable, but not (strongly) Lie metabelian.
Open question

We still don’t know $\text{dl}(U(FG))$ when $G$ is nilpotent, non-torsion, and $G'$ is cyclic $p$-group ($p = \text{char}(F)$ is odd), but $G' \neq \text{Syl}_p(G)$. All we can say about it at present is

$$\left\lceil \log_2 \left( \frac{2}{3} (|G'| + 1) \right) \right\rceil \leq \text{dl}(U(FG)) \leq \left\lceil \log_2 (|G'| + 1) \right\rceil.$$
The case $p = 2$

Write $D_k$ for the dihedral group of order $k$, and $F$ for a field of characteristic 2.

| $k$ | $|D'_k|$ | $dl_L(FD_k)$ | $dl^L(FD_k)$ | $dl(U(FD_k))$ |
|-----|---------|-------------|-------------|---------------|
| $2^3$ | $2^1$ | 2 | 2 | 2 |
| $2^4$ | $2^2$ | | | |
| $2^5$ | $2^3$ | | | |
| $2^6$ | $2^4$ | | | |
| $2^7$ | $2^5$ | | | |
| $2^8$ | $2^6$ | | | |
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|-----|---------|--------------|--------------|---------------|
| $2^3$ | $2^1$ | 2 | 2 | 2 |
| $2^4$ | $2^2$ | 2 | 2 | 2 |
| $2^5$ | $2^3$ | 2 | 2 | 2 |
| $2^6$ | $2^4$ | 2 | 2 | 2 |
| $2^7$ | $2^5$ | 2 | 2 | 2 |
| $2^8$ | $2^6$ | 2 | 2 | 2 |

Konovalov, Rossmanith, . . .:

If $\text{char}(F) = 2$ and $G$ has an abelian subgroup of index 2, then $dl_L(FG) \leq 3$. 
The case \( p = 2 \)

Write \( D_k \) for the dihedral group of order \( k \), and \( F \) for a field of characteristic 2.

| \( k \) | \( |D'_k| \) | \( dl_L(FD_k) \) | \( dl^L(FD_k) \) | \( dl(U(FD_k)) \) |
|-------|-----------|----------------|----------------|----------------|
| \( 2^3 \) | \( 2^1 \) | 2 | 2 | 2 |
| \( 2^4 \) | \( 2^2 \) | 3 | | |
| \( 2^5 \) | \( 2^3 \) | 3 | | |
| \( 2^6 \) | \( 2^4 \) | 3 | | |
| \( 2^7 \) | \( 2^5 \) | 3 | | |
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|---|---|---|---|---|
| \( 2^3 \) | \( 2^1 \) | 2 | 2 | 2 |
| \( 2^4 \) | \( 2^2 \) | 3 | | |
| \( 2^5 \) | \( 2^3 \) | 3 | | |
| \( 2^6 \) | \( 2^4 \) | 3 | | |
| \( 2^7 \) | \( 2^5 \) | 3 | | |
| \( 2^8 \) | \( 2^6 \) | 3 | | |

**Theorem (J, 2006)**

Let \( G \) be a nilpotent group with \( G' \) a finite \( p \)-group, and let \( \text{char}(F) = p \). If \( \gamma_3(G) \subseteq (G')^p \), then \( dl^L(FG) = \lceil \log_2(t(G') + 1) \rceil \).
The case \( p = 2 \)

Write \( D_k \) for the dihedral group of order \( k \), and \( F \) for a field of characteristic 2.

\[
\begin{array}{|c|c|c|c|c|}
\hline
k & |D'_k| & \text{dl}_L(FD_k) & \text{dl}^L(FD_k) & \text{dl}(U(FD_k)) \\
\hline
2^3 & 2^1 & 2 & 2 & 2 \\
2^4 & 2^2 & 3 & 3 & \\
2^5 & 2^3 & 3 & 4 & \\
2^6 & 2^4 & 3 & 5 & \\
2^7 & 2^5 & 3 & 6 & \\
2^8 & 2^6 & 3 & 7 & \\
\hline
\end{array}
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| $k$ | $|D'_k|$ | $dl_L(FD_k)$ | $dl^L(FD_k)$ | $dl(U(FD_k))$ |
|-----|---------|-------------|-------------|----------------|
| $2^3$ | $2^1$   | 2           | 2           | 2              |
| $2^4$ | $2^2$   | 3           | 3           |                |
| $2^5$ | $2^3$   | 3           | 4           |                |
| $2^6$ | $2^4$   | 3           | 5           |                |
| $2^7$ | $2^5$   | 3           | 6           |                |
| $2^8$ | $2^6$   | 3           | 7           |                |

Let’s play!
The case $p = 2$

Write $D_k$ for the dihedral group of order $k$, and $F$ for a field of characteristic 2.

| $k$ | $|D'_k|$ | $dl_L(FD_k)$ | $dl^L(FD_k)$ | $dl(U(FD_k))$ |
|-----|---------|---------------|---------------|---------------|
| $2^3$ | $2^1$   | 2             | 2             | 2             |
| $2^4$ | $2^2$   | 3             | 3             | 3             |
| $2^5$ | $2^3$   | 3             | 4             |               |
| $2^6$ | $2^4$   | 3             | 5             |               |
| $2^7$ | $2^5$   | 3             | 6             |               |
| $2^8$ | $2^6$   | 3             | 7             |               |

Let’s play!
### The case $p = 2$

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|-----|---------|-------------|-------------|--------------|
| $2^3$ | $2^1$  | 2           | 2           | 2            |
| $2^4$ | $2^2$  | 3           | 3           | 3            |
| $2^5$ | $2^3$  | 3           | 4           | 4            |
| $2^6$ | $2^4$  | 3           | 5           |              |
| $2^7$ | $2^5$  | 3           | 6           |              |
| $2^8$ | $2^6$  | 3           | 7           |              |

Let’s play!
The case $p = 2$

Write $D_k$ for the dihedral group of order $k$, and $F$ for a field of characteristic 2.

| $k$  | $|D'_k|$ | $\text{dl}_L(FD_k)$ | $\text{dl}^L(FD_k)$ | $\text{dl}(U(FD_k))$ |
|------|----------|----------------------|---------------------|---------------------|
| $2^3$| $2^1$    | 2                    | 2                   | 2                   |
| $2^4$| $2^2$    | 3                    | 3                   | 3                   |
| $2^5$| $2^3$    | 3                    | 4                   | 4                   |
| $2^6$| $2^4$    | 3                    | 5                   | 4                   |
| $2^7$| $2^5$    | 3                    | 6                   |                     |
| $2^8$| $2^6$    | 3                    | 7                   |                     |

Let’s play!
The case \( p = 2 \)

Write \( D_k \) for the dihedral group of order \( k \), and \( F \) for a field of characteristic 2.

| \( k \) | \( |D'_k| \) | \( \text{dl}_L(FD_k) \) | \( \text{dl}^*(FD_k) \) | \( \text{dl}(U(FD_k)) \) |
|---|---|---|---|---|
| \( 2^3 \) | \( 2^1 \) | 2 | 2 | 2 |
| \( 2^4 \) | \( 2^2 \) | 3 | 3 | 3 |
| \( 2^5 \) | \( 2^3 \) | 3 | 4 | 4 |
| \( 2^6 \) | \( 2^4 \) | 3 | 5 | 4 |
| \( 2^7 \) | \( 2^5 \) | 3 | 6 | 5 |
| \( 2^8 \) | \( 2^6 \) | 3 | 7 |  |
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|-----|----------|--------------|--------------|----------------|
| $2^3$ | $2^1$    | 2            | 2            | 2              |
| $2^4$ | $2^2$    | 3            | 3            | 3              |
| $2^5$ | $2^3$    | 3            | 4            | 4              |
| $2^6$ | $2^4$    | 3            | 5            | 4              |
| $2^7$ | $2^5$    | 3            | 6            | 5              |
| $2^8$ | $2^6$    | 3            | 7            | 6              |

Let’s play!
The case $p = 2$

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| $k$  | $|D'_k|$ | $dl_L(FD_k)$ | $dl^L(FD_k)$ | $dl(U(FD_k))$ |
|------|----------|---------------|--------------|----------------|
| $2^3$ | $2^1$    | 2             | 2            | 2              |
| $2^4$ | $2^2$    | 3             | 3            | 3              |
| $2^5$ | $2^3$    | 3             | 4            | 4              |
| $2^6$ | $2^4$    | 3             | 5            | 4              |
| $2^7$ | $2^5$    | 3             | 6            | 5              |
| $2^8$ | $2^6$    | 3             | 7            | 6              |

**Theorem (J-Kurdics, 2017)**

Let $\text{char}(F) = 2$, and let $G$ be a group with cyclic commutator subgroup of order $2^n$, and assume that $\text{cl}(G) = n + 1$. Then, for $n \geq 4$, we have $dl(U(FG)) < dl^L(FG) = n + 1$. 
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| $2^3$ | $2^1$    | 2            | 2            | 2             |
| $2^4$ | $2^2$    | 3            | 3            | 3             |
| $2^5$ | $2^3$    | 3            | 4            | 4             |
| $2^6$ | $2^4$    | 3            | 5            | 4             |
| $2^7$ | $2^5$    | 3            | 6            | 5             |
| $2^8$ | $2^6$    | 3            | 7            | 6             |

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Lie derived lengths vs. derived length 

Spa, 2017 12 / 14
The case $p = 2$

**Theorem (J, 2016)**

Let $\text{char}(F) = 2$, and let $G$ be a group with cyclic commutator subgroup of order $2^n$, where $n > 1$, and assume that $\text{cl}(G) \leq n$. Then either

$$\text{dl}(U(FG)) = \text{dl}^L(FG) = n + 1,$$

or

$$\text{dl}(U(FG)) = \text{dl}^L(FG) - 1 = n.$$

Furthermore, if $G' = \text{Syl}_2(G)$, then $\text{dl}(U(FG)) = n < \text{dl}^L(FG)$.

**Corollary**

For non-torsion $G$, $U(FG)$ can be metabelian, even if $G'$ is cyclic of order 4, that is when $FG$ is strongly Lie solvable, but not (strongly) Lie metabelian.
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ENGLISH IS IMPORTANT
BUT
MATH IS IMPORTANTER