Braces over a field and regular subgroups of the affine group

Francesco Catino

Università del Salento

Groups, Rings and the Yang-Baxter equation
Spa - June 23rd, 2017
The aim of the talk

The aim of this talk is to introduce the relation between these topics: braces over a field \( F \), regular subgroups of an affine group \( \text{AGL}(V) \).

Moreover, I introduce:
- recent constructions of braces over a field,
- some open problems.
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- recent constructions of braces over a field,
- some open problems.
Braces over a field

Definition

Let $V$ be a vector space over a field $F$ and let $·$ be an operation on $V$. We say that $V$ is a right brace over $F$ or right $F$-brace if

1. $(u + v) · w = u · w + v · w$,
2. $u · (v + w + v · w) = u · v + u · w + (u · v) · w$,
3. $\gamma_u: V \rightarrow V, \quad v \mapsto v · u + v$ is bijective,
4. $\mu(u · v) = (\mu u) · v$,

hold for all $u, v, w \in V$ and $\mu \in F$.

Right braces over $F$ have been introduced by Rizzo and me in (Bull. Austr. Math. Soc., 2009), where we call them circle algebras, referring to Jacobson's circle operation. The current terminology has been suggested by Rump (Note Mat., 2014).
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1. $(u + v) \cdot w = u \cdot w + v \cdot w$,
2. $u \cdot (v + w + v \cdot w) = u \cdot v + u \cdot w + (u \cdot v) \cdot w$,
3. $\gamma : V \to V, v \mapsto v \cdot u + v$ is bijective,
4. $\mu((u \cdot v)) = (\mu u) \cdot v$,

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An alternative definition

Cedó, Jespers, Okniński (Commun. Math. Phys., 2014) introduced an alternative definition of a right brace and, following this, Colazzo, Stefanelli and I (J. Algebra, 2016) have given the following alternative definition of right brace over $\mathbb{F}$.

Definition

Let $V$ be a vector space over a field $\mathbb{F}$ and let $\circ$ be an operation on $V$. We say that $V$ is a right brace over $\mathbb{F}$ or a right $\mathbb{F}$-brace if

1. $(V, \circ)$ is a group,
2. $(u + v) \circ w + w = u \circ w + v \circ w$,
3. $\mu (u \circ v) = (\mu u) \circ v + (\mu - 1)v$,

hold for all $u, v, w \in V$ and $\mu \in \mathbb{F}$.

It is easy to see that it is equivalent to the original one, considering the Jacobson’s circle operation $u \circ v := u + v + u \cdot v$, for all $u, v \in V$. 

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Examples

Example
A vector space $V$ over a field $F$ with $u \circ v := u + v$, for all $u, v \in V$, is a right $F$-brace. We call this a zero right $F$-brace.

More generally
Example
Any radical algebra $V$ over a field $F$ is a right $F$-brace (and a left $F$-brace).

Definition
A vector space $V$ over a field $F$ with an operation $\circ$ is called left $F$-brace if

1. $(V, \circ)$ is a group,
2. $u \circ (v + w) + u = u \circ v + u \circ w$,
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hold for all $u, v, w \in V$ and $\mu \in F$. 
Low dimensional cases

The unique right brace over $\mathbb{F}_q$ of dimension 1 is the field $\mathbb{F}_q$ as vector space with $u \circ v := u + v$, for all $u, v \in \mathbb{F}_q$. The description is more complicated just for dimension 2.

Example
Let $\mathbb{F}$ be a field, and let $\epsilon$ a homomorphism from the additive group of $\mathbb{F}$ to the multiplicative one. On vector space $V := \mathbb{F} \times \mathbb{F}$ we define the operation given by $(u_1, u_2) \circ (v_1, v_2) := (u_1(\epsilon(v_2)) + v_1, u_2 + v_2)$ for all $u_1, u_2, v_1, v_2 \in \mathbb{F}$. Then $V$ is a right brace over $\mathbb{F}$. 
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**Example**

Let $F$ be a field, and let $\epsilon$ a homomorphism from the additive group of $F$ to the multiplicative one.

On vector space $V := F \times F$ we define the operation given by

\[
(u_1, u_2) \circ (v_1, v_2) := (u_1(v_2 \epsilon) + v_1, u_2 + v_2)
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for all $u_1, u_2, v_1, v_2 \in F$. 

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for all $u_1, u_2, v_1, v_2 \in F$. Then $V$ is a right brace over $F$. 
The field of real numbers

Proposition

A homomorphism \( \epsilon \) from the additive group of \( \mathbb{R} \) to the multiplicative one is a function of this kind
\[
\epsilon : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto e(f(x))
\]
where \( f \) is an additive endomorphism of \( \mathbb{R} \).

Proposition

Under the assumption of continuity, the additive endomorphisms \( f \) of \( \mathbb{R} \) are the functions
\[
f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto cx
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where \( c \in \mathbb{R} \).
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**Proposition**

A homomorphism $\epsilon$ from the additive group of $\mathbb{R}$ to the multiplicative one is a function of this kind

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*If we consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$, the additive endomorphisms $f$ of $\mathbb{R}$ are the linear maps.*
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**Proposition**

If we consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$, the additive endomorphisms $f$ of $\mathbb{R}$ are the linear maps.

We know that the Hamel basis exist but we do not know how to construct them.
Right braces over $\mathbb{F}_p$

Bachiller (J. Algebra, 2015) gave a complete list of right braces of order $p^3$, for every prime $p$. So, we have the list of 3-dimensional right braces over $\mathbb{F}_p$.

We remark that all these right $\mathbb{F}_p$-braces $V$ have non-trivial socle $\text{Soc}(V) = \{a \mid a \in V, \forall v \in V a \circ v = a + v\}$, except the following one.

Example (Rump, J. Algebra 2007)

The vector space $V := \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$ with $(u_1, u_2, u_3) \circ (v_1, v_2, v_3) := (u_1 + v_1 + u_2 v_3 + u_3 (v_2 + v_1 + v_1 v_3), u_2 + v_2 + u_3 (v_1 + v_3 + v_3 v_2), u_3 + v_3)$ for all $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{F}_2$, is a right $\mathbb{F}_2$-brace with trivial socle.
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**Example (Rump, *J. Algebra* 2007)**

*The vector space* $V := \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$ *with*

$$(u_1, u_2, u_3) \circ (v_1, v_2, v_3) := (u_1 + v_1 + u_2 v_3 + u_3 (v_2 + v_1 + v_1 v_3), u_2 + v_2 + u_3 (v_1 + v_3 + v_3 v_2), u_3 + v_3)$$

*for all* $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{F}_2$, *is a right* $\mathbb{F}_2$-*brace with trivial socle.*
The affine group of a vector space

Definition

Let $V$ be a vector space over a field $F$. The affine group $AGL(V)$ of $V$ is the subgroup of $Sym(V)$ generated by the group $GL(V)$ of invertible linear maps of $V$ and the group $Tr(V)$ of translations, that is $AGL(V) := \langle GL(V), Tr(V) \rangle$.

We have:
1. $Tr(V) \trianglelefteq AGL(V)$,
2. $AGL(V) = GL(V) \cdot Tr(V)$,
3. $GL(V) \cap Tr(V) = \langle \text{id}_V \rangle$. 

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2. $AGL(V) = GL(V) \cdot Tr(V)$,
3. $GL(V) \cap Tr(V) = <id_V>$. 
A natural embedding of $AGL(V)$ into $GL(F \oplus V)$

**Proposition**

The function $\Theta : AGL(V) \rightarrow GL(F \oplus V)$, $\gamma t v \mapsto \bar{\gamma} \bar{t} v$ is an embedding.

The group $AGL(V)$ acts on the right on the set of affine vectors $\Omega := \{1 + v | v \in V\}$. 

Francesco Catino - Braces and regular subgroups
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If $\gamma \in GL(V)$, then we put

$$\bar{\gamma} : F \oplus V \rightarrow F \oplus V, \ a + u \mapsto a + (u\gamma),$$

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The group $AGL(V)\Theta$ acts on the right on the set of affine vectors

$$\Omega := \{1 + v \mid v \in V\}.$$
Finite dimensional vector spaces

Let $V$ be an $n$-dimensional vector space over a field $F$. If $(e_0, e_1, ..., e_n)$ is a basis of $F^\oplus V$ with $(e_1, ..., e_n)$ a basis of $V$, then using the previous embedding $\Theta$ we have

$$AGL(n, F) \cong \{ (1 v A) \mid v \in F^n, A \in GL(n, F) \}$$

So, we have

$$GL(n, F) \cong \{ (1 0 0 A) \mid v \in F^n, A \in GL(n, F) \}$$

and

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\[
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\[
GL(n, F) \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mid \nu \in F^n, A \in GL(n, F) \right\}
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and

\[
T_r(F^n) \cong \left\{ \begin{pmatrix} 1 & \nu \\ 0 & I_n \end{pmatrix} \mid \nu \in F^n, A \in GL(n, F) \right\}
\]
Regular subgroups of the affine group

Definition
A subgroup $T$ of $\text{AGL}(V)$ is called regular if for every $(u, v) \in V \times V$ there exists a unique $\pi \in T$ such that $u \pi = v$.

Clearly the translation group $T_r(V)$ and any of its conjugated subgroups by an element of $\text{GL}(V)$ are abelian regular subgroups of $\text{AGL}(V)$.

If $T$ is a regular subgroup of $\text{AGL}(n, F)$, then for every $v \in V$ there exists a unique element $\gamma_v \in \text{GL}(n, F)$ such that we have $(1, v)$ as first row, in the embedding of $T$ in $\text{GL}(n+1, F)$.

Thus $T = \{(1, v) \gamma_v \mid v \in F^n\}$
A subgroup $T$ of $AGL(V)$ is called regular.
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The main problem

The following problem has been explicitly stated by Liebeck, Praeger and Saxl (Mem. Amer. Soc., 2010).

Problem
Finding all regular subgroups of $AGL(V)$.

The problem has attracted the interest of many authors. For instance:
- Liebeck, Praeger, Saxl (J. Algebra, 2000)
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Caranti, Dalla Volta and Sala (2006) obtained a simple description of all abelian regular subgroups of the affine group $AGL(V)$ in terms of commutative $F$-algebras that have $V$ as underlying vector space.

Rizzo and I (2009) generalized this result obtaining a complete description of all regular subgroups of the affine group $AGL(V)$ in terms of right $F$-braces that have $V$ as underlying vector space.
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The aim of the talk

Braces

The affine group

Main results

Further results

Construction of right $F$-braces

Regular subgroups

The main problem

An approach based on braces

A link between right $F$-braces and regular subgroups

Theorem

Let $V$ be a vector space over a field $F$.

Denote by $RB$ the class of right $F$-braces with underlying vector space $V$ and by $T$ the set of all regular subgroups of the affine group $AGL(V)$.

1. If $V \in RB$, then $T(V) = \{ \tau_x | x \in V \}$, where $\tau_x : V \rightarrow V$, $y \mapsto y \circ x$, is a regular subgroup of the affine group $AGL(V)$.

2. The function $f : RB \rightarrow T$, $V \mapsto T(V)$, is a bijection. Moreover, in this correspondence isomorphic right $F$-braces correspond regular subgroups of $AGL(V)$ conjugated under the action of $GL(V)$. 
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**Let $V$ be a vector space over a field $F$. Denote by $\mathcal{RB}$ the class of right $F$-braces with underlying vector space $V$ and by $\mathcal{T}$ the set of all regular subgroups of the affine group $\text{AGL}(V)$.**

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Moreover, in this correspondence isomorphic right $F$-braces correspond regular subgroups of $AGL(V)$ conjugated under the action of $GL(V)$. 
Some useful remarks
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where $M_v$ is the matrix associated to the linear map

$$\gamma_v : V \rightarrow V, \quad x \mapsto x \circ v - v,$$

for every $v \in V$. 
An example

Let $F$ be a field and $\alpha$ an endomorphism of the additive group of $F$.

On the vector space $V := F \times F$ we define the operation given by

$$(u_1, u_2) \circ (v_1, v_2) := (u_1 + v_1, u_1(\alpha(v_1)) + u_2 + v_2)$$

for all $u_1, u_2, v_1, v_2 \in F$.

Then $V$ is a radical $F$-algebra and so it is right $F$-brace.

Then, we have

$$\gamma((v_1, v_2)) = (1, 0) \circ (v_1, v_2) - (v_1, v_2) = (1, v_1 \alpha)$$

and so $T(V) =$ \begin{align*}
\begin{cases}
\begin{pmatrix}
1 & v_1 \\
0 & 1
\end{pmatrix} & \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\end{cases}
\end{align*}

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$$
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$$

for all $v_1, v_2 \in F$, and so $T(V) = \{ 
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for all $v_1, v_2 \in F$, and so

$$T(V) = \left\{ \begin{pmatrix} 1 & v_1 & v_2 \\ 0 & 1 & v_1\alpha \\ 0 & 0 & 1 \end{pmatrix} \middle| v_1, v_2 \in F \right\},$$

is a regular subgroup of $AGL(2, F)$. 
### A further example

Let $F$ be a field and $\epsilon$ a homomorphism from the additive group of $F$ to the multiplicative one. It is easy to see that the group $T = \{ \begin{pmatrix} 1 & v_1 \\ v_2 \\ 0 & 1 \end{pmatrix} : v_1, v_2 \in F \}$ is a regular subgroup of the affine group $AGL(2, F)$. By the previous result, $T$ is in correspondence with the right $F$-brace over the vector space $V := F^2$ where the multiplication is given by $(u_1, u_2) \circ (v_1, v_2) = (u_1, u_2)(v_2 \epsilon) + (v_1, v_2)$ for all $u_1, u_2, v_1, v_2 \in F$. 

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By the previous result, $T$ is in correspondence with the right $F$-brace over the vector space $V := F^2$. 
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It is easy to see that the group

$$T = \left\{ \begin{pmatrix} 1 & v_1 & v_2 \\ 0 & v_2 \epsilon & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| v_1, v_2 \in F \right\}$$

is a regular subgroup of the affine group $AGL(2, F)$.

By the previous result, $T$ is in correspondence with the right $F$-brace over the vector space $V := F^2$ where the multiplication is given by

$$(u_1, u_2) \circ (v_1, v_2) = (u_1, u_2) \begin{pmatrix} v_2 \epsilon & 1 \\ 0 & 1 \end{pmatrix} + (v_1, v_2) = (u_1 (v_2 \epsilon) + v_1, u_2 + v_2)$$

for all $u_1, u_2, v_1, v_2 \in F$. 
The aim of the talk
Braces
The affine group
Main results
Further results
Construction of right $F$-braces

The intersection with the translation subgroup
Hegedűs' result
Using right $F$-braces

The problem of the existence of regular subgroups of the affine group $AGL(n, F)$ having only the identity as translation was stated by Liebeck, Praeger, Saxl (2000).

Problem
In which cases are there regular subgroups of the affine group $AGL(n, F)$ having only the identity as translation?

They also proved that no such regular subgroup exists in $AGL(2, F_p)$, $AGL(3, F_p)$, $p > 3$, $AGL(4, F_2)$.
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*The affine group* $AGL(n, \mathbb{F}_p)$ *has a regular subgroup which contains only the identity as translation if*

- $p = 2$ and $n = 3$, or $n \geq 5$,
- $p \neq 2$ and $n \geq 4$.  


---

**The intersection with the translation subgroup**

**Hegedűs’ result**

Using right $F$-braces
Hegedűs’ regular subgroups

Hegedűs’ result

Using right $F$-braces

In the cases $p = 2$ and $n \geq 3$, odd $p \neq 2$ and $n \geq 4$, Hegedűs provides concrete regular subgroups in the following way:

Let $q$ be a non-degenerate quadratic form on $F_{n-1}$, let $X$ be the matrix of the polar form of $q$, let $A$ be a matrix of order $p$ in the orthogonal group associated with $q$.

Then the set

$$T := \left\{ \begin{pmatrix} 1 & w & \sum_{k=0}^{p} k w A \end{pmatrix} \bigg| \begin{array}{c} k \in F_p, \ w \in F_{n-1} \end{array} \right\}$$

is a regular subgroup of the affine group $\text{AGL}(n, F_p)$ containing only the trivial translation.
Hegedűs’ regular subgroups

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Then the set
\[
T := \left\{ \begin{pmatrix} 1 & wq + k & w \\ 0 & 1 & 0 \\ 0 & A^k Xw^T & A^k \end{pmatrix} \bigg| k \in \mathbb{F}_p, w \in \mathbb{F}_p^{n-1} \right\}.
\]

is a regular subgroup of the affine group \( AGL(n, \mathbb{F}_p) \) containing only the trivial translation.
An example

Let $V = F_2 \times F_2 \times F_2$ and let $q$ be the non-degenerate quadratic form on $F_2^2$, given by $(w_2, w_3) q = w_2 w_3$, and let $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the matrix $X$ is equal to $A$, and the Hegedüs' regular subgroup related to $q$ and $A$ is $T = \begin{cases} \begin{pmatrix} 1 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ 0 & 1 & 0 & 0 \\ 0 & \mathbf{v}_3 & 1 & 0 \\ \mathbf{v}_2 & \mathbf{v}_1 & 0 & 1 \end{pmatrix} & | \mathbf{v}_1, \mathbf{v}_2 \in F_2 \end{cases}$.

Note that the right $F_2$-brace associated to $T$ is the Rump's brace seen before.
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$$
T = \left\{ \begin{pmatrix}
1 & v_1 & v_2 & v_3 \\
0 & 1 & 0 & 0 \\
0 & v_3 & 1 & 0 \\
0 & v_2 + v_1 + v_1 v_3 & v_1 + v_3 + v_3 v_2 & 1
\end{pmatrix} \mid v_1, v_2 \in \mathbb{F}_2 \right\}
$$

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Using right $F$-braces

We can describe the intersection of a regular subgroup with the group of translations by $F$-braces.

Proposition (Rizzo and I, 2009) If $V$ is a right $F$-brace, $\mathcal{T}(V) = \{\tau_v | v \in V\}$, where $\tau_v: V \to V, u \mapsto u \circ v$, for every $v \in V$, and $\mathcal{T}_r(V)$ is the translation group, then $\mathcal{T}(V) \cap \mathcal{T}_r(V) = \{\tau_a | a \in \text{Soc}(V)\}$, where we recall that the socle of $V$ is $\text{Soc}(V) := \{a | a \in V, \forall v \in V a \circ v = a + v\}$. 

Francesco Catino - Braces and regular subgroups
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If $V$ is a right $F$-brace, $T(V) = \{ \tau_v \mid v \in V \}$, where $\tau_v : V \to V$, $u \mapsto u \circ v$, for every $v \in V$. 

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A generalization of the the Hegedűs’ result

The affine group $A\text{GL}(n,F)$ has a regular subgroup which contains only the identity as translation if:

- $p$ odd, $m = 1$ and $m \geq 3$;
- $p$ odd, $m > 1$ and $m \geq 4$;
- $p = 2$, $m = 1$ and $n \geq 3$, or $n \geq 5$;
- $p = 2$, $m > 1$ and $n = 4$, $n = 5$, or $n = 6$ and $n \geq 8$.

The main tool is the asymmetric product of right $F$-braces.

In this way, Colazzo, Stefanelli and I (2016) generalized the Hegedűs’ result.

Proposition

The affine group $A\text{GL}(n,F)$ has a regular subgroup which contains only the identity as translation if $p$ odd, $m = 1$ and $m \geq 3$; $p$ odd, $m > 1$ and $m \geq 4$; $p = 2$, $m = 1$ and $n \geq 3$, or $n \geq 5$; $p = 2$, $m > 1$ and $n = 4$, $n = 5$, or $n = 6$ and $n \geq 8$.

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The affine group $\operatorname{AGL}(n, \mathbb{F}_p^m)$ has a regular subgroup which contains only the identity as translation if

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- $\alpha : S \to \text{Aut}(T)$ be a homomorphism of the multiplicative group of $S$ into the group of automorphisms of the right $F$-brace $T$.

If we set $t^s := t(s\alpha)$ and
\[
(t_1, t_2)b \circ s + ((t_1 + t_2)^s \circ t_3, t_3)b = (t_1^s \circ t_3, t_2^s \circ t_3)b + s
\]
holds for all $s \in S$ and $t_1, t_2, t_3 \in T$,
Let

- $F$ be a field of characteristic $p \neq 2$;
- $S$ and $T$ be right $F$-braces;
- $b : T \times T \to S$ be a bilinear and symmetric map;
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holds for all $s \in S$ and $t_1, t_2, t_3 \in T$, then we may define over $S \times T$ a structure of right $F$-brace.
The asymmetric product of right $F$-braces

Namely, we may set the addition, the multiplication over $S \times T$ and the scalar multiplication
\[
(s_1, t_1) + (s_2, t_2) = (s_1 + s_2, (t_1, t_2)b, t_1 + t_2)
\]
\[
(s_1, t_1) \circ (s_2, t_2) = (s_1 \circ s_2, ts_2 \circ t_1)
\]
\[
\mu(s_1, t_1) = (\mu s_1 + \mu^2 (\mu^{-1})^2 (t_1, t_2)b, \mu t_1)
\]
for all $s_1, s_2 \in S$, $t_1, t_2 \in T$ and $\mu \in F$.

This right $F$-brace is called the asymmetric product of $S$ by $T$.
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$$(s_1, t_1) \circ (s_2, t_2) = (s_1 \circ s_2, \ t_1 s_2 \circ t_2)$$

$$\mu(s_1, t_1) = (\mu s_1 + \frac{\mu(\mu - 1)}{2}(t_1, t_2)b, \ \mu t_1)$$
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(s_1, t_1) \circ (s_2, t_2) = (s_1 \circ s_2, \ t_1^{s_2} \circ t_2)
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\mu(s_1, t_1) = (\mu s_1 + \frac{\mu(\mu - 1)}{2}(t_1, t_2)b, \ \mu t_1)
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for all $s_1, s_2 \in S$, $t_1, t_2 \in T$ and $\mu \in F$. 

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for all $s_1, s_2 \in S$, $t_1, t_2 \in T$ and $\mu \in F$. This right $F$-brace is called the asymmetric product of $S$ by $T$. 
In the case of characteristic 2, we have only a partial result.
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- $\alpha : F \rightarrow \text{Aut}(T)$ be a homomorphism of the multiplicative group of $F$ into the group of automorphisms of the zero $F$-brace $T$. 
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If

$$(t_1^s \circ t_2)b = (t_1)q \circ s + (t_2)q + (t_1^s \circ t_2)q$$

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Characteristic 2

Discussion of the asymmetric product of right $F$-braces and further results on regular subgroups.
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\[(s_1, t_1) + (s_2, t_2) = (s_1 + s_2 + (t_1, t_2)b, t_1 + t_2)\]
\[(s_1, t_1) \circ (s_2, t_2) = (s_1 \circ s_2, t_1^s_2 \circ t_2)\]
\[\mu(s_1, t_1) = (\mu s_1 + \mu(\mu - 1)(t_1)q, \mu t_1)\]

for all $s_1, s_2 \in F$, $t_1, t_2 \in T$ and $\mu \in F$. 
Particular cases

We extend Hegedűs' result by the asymmetric product of zero $F$-braces.

Let us note that if $S$ and $T$ are zero right braces over a field $F$ of odd characteristic the condition on $b$ and $\alpha$ becomes easier:

$$(t_1, t_2) b = (t_{s_1}, t_{s_2}) b,$$ for all $t_1, t_2 \in T$ and $s \in S$.

In a similar way, if $T$ is a zero brace over a field $F$ of characteristic 2, the condition on $q$ and $\alpha$ becomes easier:

$$(ts) q = (t) q,$$ for all $t \in T$ and $s \in F$. 

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Pellegrini and Tamburini (2017) generalized the result of Colazzo, Stefanelli and me (2016) proving the following.

Theorem
Let $F$ be any field and let $W$ be a subspace of $F^n$ as a vector space over its prime field $F_0$. Assume that one of the following conditions hold:
- $n = 3$ and $F = F_2$;
- $n \geq 4$ and $\text{char } F \neq 2$;
- $n \geq 5$ and $\text{char } F = 2$;

Then, there exists a regular subgroup $R_W$ of $\text{AGL}(n, F)$ such that $R_W \cap \text{Tr}(F^n) \cong (W, +)$.

In particular there exists a regular subgroup $R$ that contains only the identity as translation.

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The key lemma

Lemma

Let $m, n \in \mathbb{N}$ and let $d$ be a fixed row vector of $\mathbb{F}^n$.

Let $q$ be a quadratic form on $\mathbb{F}^m$ with polar form $X$ and let $\varphi$ be a group homomorphism from $\left(\mathbb{F}^n, +\right)$ into the orthogonal group $O_m(\mathbb{F}, q)$.

Then $R = \left\{ \begin{array}{l}
\begin{pmatrix}
1 & (v_q) + k w \\
0 & 0
\end{pmatrix}
\otimes a \varphi(X v^T) \\
\end{array} \right| w \in \mathbb{F}^m, k \in \mathbb{F}^n \right\}$ is a regular subgroup of $AGL(m+n, \mathbb{F})$ and, if $q$ is non-degenerate and $d \neq 0$, we have $R \cap Tr(\mathbb{F}^m + k) \sim \text{Ker}(\varphi)$. 

Francesco Catino - Braces and regular subgroups
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R = \left\{ \begin{pmatrix} 1 & (vq)d + k & w \\ 0 & I_n & 0 \\ 0 & (a \varphi)Xv^T \otimes d & a \varphi \end{pmatrix} \middle| \ w \in F^m, k \in F^n \right\}
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Then we can obtain right $F$-braces with trivial socle.