Simply connected Latin quandles

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1 Introduction

Quandles are groupoids introduced by Joyce [4] as algebraic objects that can be used to obtain invariants of oriented knots ([8]). We are interested in extension theory of quandles, with a particular focus on a special class of extensions of connected quandles. A connected quandle Y is a covering of a quandle X if Y is a factor of Y by a congruence of Y such that refines the congruence of X. where (x, y) ∈ Lm(Y) if and only if the left translations Lm x, Lm y on Y coincide. It turns out that the structure of a covering of X is described by the so-called (constant) quandle cocyle. Connected quandles which admits the trivial cocyle are called simply connected. We develop a combinatorial approach to latin quandle cocyles in order to show the following result:

Theorem 1. Let X be a finite connected quandle that is affine over a cyclic group, or a finite doubly transitive quandle of size different from 3. Then X is simply connected.

i.e. quandles belonging to such classes have just trivial coverings (extending a result given in [5] for quandles of prime size). Moreover, using a categorical characterization in [2], we give an alternative characterization of finite simply connected quandles.

2 Quandles

Definition 1. A groupoid (X, ·) is a quandle if the map Lx : y → x · y is a bijection of X for every x ∈ X and

where α is a congruence of X.

Example 1. Let S be a set. Then P2(S, S) is a quandle where x · y = xy for all x, y ∈ S.

Example 2. Let G be a group and be a subset of G closed under conjugation. Then ConG(H) = {h ∈ H | for each g ∈ G, hgh−1 = g}, is a called conjugation quandle on H. Moreover, ConG(G) acts transitively on X, and doubly transitive if ConG(X) acts doubly transitively on X. The group Inv(X) = (LX) is called degree of group action on X.

3 Quandle coverings

Definition 2. A connected quandle Y is a covering of a quandle X if X ≈ Y/α, where α is a congruence of Y that refines (1).

Definition 3. [1] Let S be a quandle and S be a non-empty set. A (constant) quandle cocyle β ∈ G2(S, X) is a map β : X × X → Hom(S, S) such that

where α is a congruence of X.

The set of all quandle cocycles X × X → Hom(S, S) of the form (3).

Proposition 1. [1] Let X be a quandle and S be a non-empty set. Then X is simply connected if and only if X ≈ X/β, where β ∈ Z2(S, X).

4 Characterization of simply connected quandles

Definition 5. A connected quandle X is simply connected if every covering of X is trivial up to isomorphism, that is, if H1(X, S) = {1} for every non-empty set S, i.e. X admits only trivial coverings.

Proposition 3. [1] The assignment Adj : Gnd → Grps X → Adj(X) = (X | S), where X is a subquandle of X and X = X, acts transitively on X, and doubly transitive if acts doubly transitively on X. The group Inv(X) = (LX) is a quandle and the map L−1 : X → L−1 is a quandle morphism.

5 Normalized Cocycles for latin quandles

Definition 6. A quandle is latin if , and is a bijection for every x ∈ X. Let u ∈ U. Then β ∈ Z2(U, X) is said to be u-normalized if −1 for every x ∈ X.

Proposition 4. Let X be a latin quandle. Then X is a non-empty set, x ∈ X, and X ≈ X/β, where β ∈ Z2(U, X).

Remark 1. Note that for every , and if only if x = y and , and then preserves the property (x) = (y). On the other hand, if , and if only if x = y and . Moreover, if , then .

Proposition 5. Let X be a latin quandle and Cβ ≈ (α(x, u) | x β ∈ Z2(U, X)) is a special (affine) quandle, the size of the and β orbits can be computed in terms of the number of cycles of α and the order of the elements of A.

6 Simply connected latin quandles

For a latin quandle X, X × X splits into orbits Oi on which (f, h) by Lemma 4, the set Cβ splits into orbits to Oi, Cβi, Cβ2, and their complement in Cβ. If X = (X, α) is a principal (affine) quandle, the size of the and β orbits can be computed in terms of the number of cycles of α and the order of the elements of A.

Theorem 6. Let X be a finite latin quandle. Then X is simply connected.

Proof. In the doubly transitive case, there are at most five orbits of (f, h), namely (h, a), (h, b), (h, c), (h, d) and certain sets of Oi. Using the cocycle condition and the action of h, β is a one-to-one correspondence.

Proposition 6. Let X be a Latin quandle, the size of a is one and if then and are conjugate.

6.2 Affine quandles over cyclic groups

Graha showed:

Proposition 7. Let be a prime and X = (Zp, αn) an affine quandle. Then X is simply connected.

Proof. Let X = (Z, α) be a latin affine quandle and let be a u-normalized cocycle. Then (x, y) = (x + y, x) for every integer x and every x, y ∈ X.

Proposition 8. Let X = (Z, α) be an affine quandle and let be a u-normalized cocyle. Then (x, y) = (x + y, x) for every x, y ∈ X.

Remark 2. Note that for every x ∈ X and if only if and preserves the property (x) = (y). On the other hand, if and if only if and . Moreover, if , then .

For example, if (f, h) acts on the set Cβ = Oi(x, y) times y ∈ S of (f, h) orbits.

Lemma 4. Let X be a latin quandle and Cβi = (α(x, u) | x β, u ∈ X) are u-normalized cocyles, then

and let G = (f, g)(h, j) (h, j)(g, k) for every x, y ∈ X and every y ∈ X.

References