An element of a(n associative) ring (with 1) is clean if it is the sum of a unit and an idempotent. A ring is clean if every element in it is clean. The concept of clean rings was formulated by Nicholson [8] in the course of his study of exchange rings, for both are closely related. Cleanliness in group rings has been studied from 2001 [3] and has been attracting attention ever since.

Several related concepts have been proposed. In 2010, Vaš proposed the definition of a \( \ast \)-clean ring ("star"-clean) [9]: a \( \ast \)-ring (ring with an involution \( \ast \)) in which every element may be written as a sum of a unit and a projection (a \( \ast \)-symmetric idempotent). Clearly, every \( \ast \)-clean ring is clean. So Vaš asked in [9]: when is a \( \ast \)-ring clean, but not \( \ast \)-clean?

Since every group \( G \) is endowed with the classical involution \( g \mapsto g^{-1} \), group rings \( RG \) are almost always \( \ast \)-rings: if \( R \) is a commutative rings, for instance, an involution in \( RG \) is obtained from the \( R \)-linear extension of the classical involution in \( G \) (and is also called the classical involution in \( RG \)). The \( \ast \)-cleanliness of group rings was first approached in 2011 [6]. Very little is still known about conditions under which a group ring with the classical involution is \( \ast \)-clean.

In this talk, we present clean rings, \( \ast \)-clean rings, some answers to Vaš’s question, their story in the realm of group rings, and some recent results [1, 2, 4, 5, 7].

References