n-UJ Rings

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Objectives

The aim of this study:

- Firstly, is to give the general description of the n-UJ rings and mention the importance of their usage.
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- Moreover, define the rings $R$ for which $u - u^n$ belongs to the Jacobson radical for all units $u$ of $R$, where $n \geq 1$ is a fixed integer.
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- Moreover, define the rings \( R \) for which \( u - u^n \) belongs to the Jacobson radical for all units \( u \) of \( R \), where \( n \geq 1 \) is a fixed integer.
- Also, study the n-UJ property under some algebraic construction, the trivial extension.
Objectives

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- Firstly, is to give the general description of the n-UJ rings and mention the importance of their usage.
- Moreover, define the rings $R$ for which $u - u^n$ belongs to the Jacobson radical for all units $u$ of $R$, where $n \geq 1$ is a fixed integer.
- Also, study the n-UJ property under some algebraic construction, the trivial extension.
- Finally, obtain the Morita context of n-UJ rings.
In 2015, Danchev defined \( UU \) rings such that a ring \( R \) is called \( UU \) if \( 1 + N(R) = U(R) \) [1].
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In the following years, Leroy etc. defined *UJ* rings such that a ring $R$ is called *UJ* if $1 + J(R) = U(R)$ [3].
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In 2015, Danchev defined $UU$ rings such that a ring $R$ is called $UU$ if $1 + N(R) = U(R)$ [1].

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Then we observed that all $UJ$ rings are $n-UJ$ and every $n-UJ$ ring is $\infty - UJ$.

Also the behavior of $n-UJ$ property under some classical ring construction, the trivial extension and the Morita context can be studied.
**General Properties of n-UJ rings**

**Definition 2.1**

Let $n \in \mathbb{N}$. A ring $R$ is said to be an n-UJ ring if $u - u^n \in J(R)$ for each $u \in U(R)$.
General Properties of $n$-UJ rings

**Definition 2.1**
Let $n \in \mathbb{N}$. A ring $R$ is said to be an $n$-UJ ring if $u - u^n \in J(R)$ for each $u \in U(R)$.

**Definition 2.2**
Let $n \in \mathbb{N} \cup \{\infty\}$. A ring $R$ is said to be an $\infty$-UJ ring if for each $u \in U(R)$ there exists $n$ such that $u - u^n \in J(R)$.
General Properties of $n$-UJ rings

Definition 2.3

For $n \in \mathbb{N}$, consider the following sets:

$$
U_n(R) = \{ u^{n-1} : u \in U(R) \} \subseteq U(R)
$$

$$
V_n(R) = \{ u \in U(R) : u^{n-1} \in 1 + J(R) \}
$$
General Properties of $n$-UJ rings

Lemma 2.3

The following statements are equivalent for a ring $R$ and $n \in \mathbb{N}$:

1. $R$ is an $n$-UJ ring;
2. $\mathbb{V}_n(R) = U(R)$.
3. $U_n(R) \subseteq 1 + J(R)$
4. $U(R/J(R)) = \{\bar{u} \in R/J(R) : \bar{u}^{n-1} = \bar{1}\} = \mathbb{V}_n(R/J(R))$.
General Properties of \(n\)-UJ rings

**Proposition 2.4.**

If \(R\) is an \(n\)-UJ ring with \(n - 1 \in U(R)\), then \(N(R) \subseteq J(R)\).
General Properties of $n$-UJ rings

Proposition 2.4.
If $R$ is an $n$-UJ ring with $n - 1 \in U(R)$, then $N(R) \subseteq J(R)$.

Proposition 2.5.
For a ring $R$, the following observations hold:

1. Let $I \subseteq J(R)$ be an ideal of $R$. Then $R$ is an $n$-UJ ring if and only if $R/I$ is an $n$-UJ ring.

2. Let $R$ be an $n$-UJ ring and $T$ a subring of $R$. Then $T$ is an $n$-UJ ring if $T \cap J(R) \subseteq J(T)$.
General Properties of n-UJ rings

Proposition 2.6.
If \( n \in \mathbb{N} \cup \{\infty\} \) and \( R \) is an n-UJ ring, then \( eRe \) is n-UJ for any \( e^2 = e \in R \).
Proposition 2.6.

If $n \in \mathbb{N} \cup \{\infty\}$ and $R$ is an n-UJ ring, then $eRe$ is n-UJ for any $e^2 = e \in R$.

Proposition 2.7.

If $R$ is an n-UJ ring and $n - 1 \in U(R)$, then $R/J(R)$ is reduced and so is abelian.
**General Properties of n-UJ rings**

**Example 2.8**

Let \( R = \mathbb{Z}_2 < x, y > / (x^2) \), where \( \mathbb{Z}_2 \) is an algebra generated by \( x \) and \( y \). Then \( N(R) = \mathbb{Z}_2 x + xRx \), \( U(R) = 1 + \mathbb{Z}_2 x + xRx \) and \( J(R) = 0 \). Since \( (U(R))^2 = 1 \), we obtain that \( R \) is a 3-UJ ring, but \( R/J(R) \cong R \) is not reduced.
General Properties of $n$-$UJ$ rings

**Theorem 2.9**

The following conditions are equivalent for a ring $R$:

1. $R$ is a $UJ$-ring.
2. There exists $k$ such that $R$ is $(2^k + 1)$-$UJ$, $R/J(R)$ is reduced and $2 \in J(R)$. 
The Trivial Extension and the Morita Context

**Definition 3.1**

Let $R$ be a ring and $M$ a bimodule over $R$. The trivial extension of $R$ and $M$ is

$$T(R, M) = \{(r, m) : r \in R \text{ and } m \in M\}$$

with an addition defined componentwise and a multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

The trivial extension $T(R, M)$ is isomorphic to the subring

$$\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R \text{ and } m \in M$$

of the formal $2 \times 2$ matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ and also $T(R, R) \cong R[x]/(x^2)$. We also note that the set of units of trivial extension $T(R, M)$ is

$$U(T(R, M)) = T(U(R), M)$$
The Trivial Extension and the Morita Context

Definition 3.2

A Morita context is a 4-tuple \( \begin{pmatrix} A & M \\ N & B \end{pmatrix} \), where \( A \) and \( B \) are rings, \( A M_B \) and \( B N_A \) are bimodules, and there exist context products \( M \times N \rightarrow A \) and \( N \times M \rightarrow B \) written multiplicatively as \((w, z) = wz\) and \((z, w) = zw\), such that \( \begin{pmatrix} A & M \\ N & B \end{pmatrix} \) is an associative ring with the obvious matrix operations.
A Morita context \( \begin{pmatrix} A & M \\ N & B \end{pmatrix} \) is called trivial if the context products are trivial, i.e., \( MN = 0 \) and \( NM = 0 \). We have

\[
\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N),
\]

where \( \begin{pmatrix} A & M \\ N & B \end{pmatrix} \) is a trivial Morita context by [3].
The Trivial Extension and the Morita Context

**Theorem 3.4**

Let $M$ be an $(R, R)$ bimodule. The trivial extension $R$ is an $n$-$UJ$ ring if and only if $T(R, M)$ is an $n$-$UJ$ ring.
The Trivial Extension and the Morita Context

Theorem 3.4
Let $M$ be an $(R, R)$ bimodule. The trivial extension $R$ is an $n$-$UJ$ ring if and only if $T(R, M)$ is an $n$-$UJ$ ring.

Corollary 3.5
Let $M$ be an $(R, S)$ bimodule. Then the formal triangular matrix ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is an $n$-$UJ$ ring if and only if $R$ and $S$ are $n$-$UJ$ rings.
Two Examples: The Dorroh Extension and the Tail Ring Extension

**Definition 4.1**

If $T$ is a non-unital ring, its Dorroh extension is the unital ring $\mathbb{Z} \oplus T$, with component-wise addition, and multiplication given by

$$(n_1, t_1)(n_2, t_2) = (n_1 n_2, t_1 t_2 + n_1 t_2 + n_2 t_1).$$
Example 4.2

Let $T$ be a non-unital ring. If the Dorroh extension $Z \oplus T$ is an $n$-UJ ring, then $T$ is an $n$-UJ ring.

Let $u \in U(T)$. Then $(0, u) \in U(Z \oplus T)$. By the hypothesis, $(0, u) - (0, u^n) = (0, u - u^n) \in J(Z \oplus T)$. But $J(Z \oplus T) = J(Z) \oplus J(T)$ which implies $u - u^n \in J(T)$ since $J(Z) = 0$. 
Two Examples: The Dorroh Extension and the Tail Ring Extension

Example 4.3

$\mathbb{R}[D, C]$ is an n-UJ ring if and only if $D$ and $C$ are n-UJ rings.
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Many Thanks!

Thank you all for your patience and attentions