SKEW LATTICES AND SET-THEORETIC
SOLUTIONS OF THE YANG-BAXTER
EQUATION

(joint work with Karin Cvetko-Vah)

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Groups, rings and associated structures
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VUB VRIJE UNIVERSITEIT BRUSSEL
A **set-theoretic solution** of the Yang-Baxter equation is a pair $(X, r)$ such that $X$ is a non-empty set and

$r : X \times X \to X \times X : (x, y) \mapsto (\sigma_x(y), \gamma_y(x))$ is a map where

$$(r \times id_X) \circ (id_X \times r) \circ (r \times id_X) = (id_X \times r) \circ (r \times id_X) \circ (id_X \times r).$$
A **set-theoretic solution** of the Yang-Baxter equation is a pair \((X, r)\) such that \(X\) is a non-empty set and \(r : X \times X \to X \times X : (x, y) \mapsto (\sigma_x(y), \gamma_y(x))\) is a map where

\[
(r \times \text{id}_X) \circ (\text{id}_X \times r) \circ (r \times \text{id}_X) = (\text{id}_X \times r) \circ (r \times \text{id}_X) \circ (\text{id}_X \times r).
\]

- **Left non-degenerate**: \(\sigma_x\) is bijective, for all \(x \in X\)

- **Right non-degenerate**: \(\gamma_x\) is bijective, for all \(x \in X\)

- **Involutive**: \(r = \text{id}_X\)

- **Idempotent**: \(r \circ r = r\)

- **Cubic**: \(r \circ r \circ r = r\)
A set-theoretic solution of the Yang-Baxter equation is a pair $(X, r)$ such that $X$ is a non-empty set and $r : X \times X \to X \times X : (x, y) \mapsto (\sigma_x(y), \gamma_y(x))$ is a map where

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SET-THEORETIC SOLUTIONS

Definition

A **set-theoretic solution** of the Yang-Baxter equation is a pair $(X, r)$ such that $X$ is a non-empty set and $r : X \times X \to X \times X : (x, y) \mapsto (\sigma_x(y), \gamma_y(x))$ is a map where

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- **Left non-degenerate**: $\sigma_x$ is bijective, for all $x \in X$
- **Right non-degenerate**: $\gamma_x$ is bijective, for all $x \in X$
- **Involutive**: $r^2 = id_X$
A **set-theoretic solution** of the Yang-Baxter equation is a pair \((X, r)\) such that \(X\) is a non-empty set and \(r : X \times X \to X \times X : (x, y) \mapsto (\sigma_x(y), \gamma_y(x))\) is a map where

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A **set-theoretic solution** of the Yang-Baxter equation is a pair $(X,r)$ such that $X$ is a non-empty set and $r : X \times X \to X \times X : (x,y) \mapsto (\sigma_x(y),\gamma_y(x))$ is a map where

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- **Right non-degenerate**: $\gamma_x$ is bijective, for all $x \in X$
- **Involutive**: $r^2 = id_X$
- **Idempotent**: $r^2 = r$
- **Cubic**: $r^3 = r$
A skew lattice (SL) is a set \( S \) endowed with a pair of idempotent and associative operations \( \wedge \) and \( \lor \) which satisfy the absorption laws

\[
x \wedge (x \lor y) = x = x \lor (x \wedge y) \quad \text{and} \quad (x \wedge y) \lor y = y = (x \lor y) \wedge y.
\]

**Notation:** \((S, \wedge, \lor)\)
SKEW LATTICES

Definition

A **skew lattice** (SL) is a set $S$ endowed with a pair of idempotent and associative operations $\land$ and $\lor$ which satisfy the absorption laws

$$x \land (x \lor y) = x = x \lor (x \land y) \quad \text{and} \quad (x \land y) \lor y = y = (x \lor y) \land y.$$  

**Notation:** $(S, \land, \lor)$

Examples

- Lattices
A skew lattice (SL) is a set $S$ endowed with a pair of idempotent and associative operations $\land$ and $\lor$ which satisfy the absorption laws

$$x \land (x \lor y) = x = x \lor (x \land y) \quad \text{and} \quad (x \land y) \lor y = y = (x \lor y) \land y.$$  

**Notation:** $(S, \land, \lor)$

**Examples**
- Lattices
- $(\{0, 1, 2\}, \land, \lor)$, where

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A skew lattice \((S, \wedge, \vee)\) is called a **strong distributive solution** of the Yang-Baxter equation if \((S, r)\) is a set-theoretic solution of the Yang-Baxter equation, where

\[
r : S \times S \to S \times S : (x, y) \mapsto (x \wedge y, x \vee y).
\]

**Remark:** \((S, r)\) is cubic
STRONG DISTRIBUTIVE SOLUTIONS

{Strongly and co-strongly distributive SL} 
\[\Rightarrow\] 
{Strong distributive solution} 
\[\Rightarrow\] 
{Distributive and cancellative SL}
STRONG DISTRIBUTIVE SOLUTIONS

\{
\text{Strongly and co-strongly distributive SL}\}

\uparrow

\{
\text{Strong distributive solution}\}

\uparrow

\{
\text{Distributive and cancellative SL}\}

**Strongly distributive:**
\[
x \land (y \lor z) = (x \land y) \lor (x \land z)
\]
\[
(x \lor y) \land z = (x \land z) \lor (y \land z)
\]
STRONG DISTRIBUTIVE SOLUTIONS

{Strongly and co-strongly distributive SL}

\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \]
\[ (x \lor y) \land z = (x \land z) \lor (y \land z) \]

Strongly distributive:

Co-strongly distributive:

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \]
\[ (x \land y) \lor z = (x \lor z) \land (y \lor z) \]

{Strong distributive solution}

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \]
\[ (x \land y) \lor z = (x \lor z) \land (y \lor z) \]

{Distributive and cancellative SL}

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \]
\[ (x \land y) \lor z = (x \lor z) \land (y \lor z) \]
STRONG DISTRIBUTIVE SOLUTIONS

{Strongly and co-strongly distributive SL}

{Strong distributive solution}

{Distributive and cancellative SL}

**Strongly distributive:**

\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \]

\[ (x \lor y) \land z = (x \land z) \lor (y \land z) \]

**Co-strongly distributive:**

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \]

\[ (x \lor y) \land z = (x \land z) \lor (y \lor z) \]

**Distributive:**

\[ x \land (y \lor z) \land x = (x \land y \land x) \lor (x \land z \land x) \]

\[ x \lor (y \land z) \lor x = (x \lor y \lor x) \land (x \lor z \lor x) \]
STRONG DISTRIBUTIVE SOLUTIONS

\{\text{Strongly and co-strongly distributive SL}\}

\{\text{Strong distributive solution}\}

\{\text{Distributive and cancellative SL}\}

\textbf{Strongly distributive:}\quad x \land (y \lor z) = (x \land y) \lor (x \land z)
\[
(x \lor y) \land z = (x \land z) \lor (y \land z)
\]

\textbf{Co-strongly distributive:}\quad x \lor (y \land z) = (x \lor y) \land (x \lor z)
\[
(x \land y) \lor z = (x \lor z) \land (y \lor z)
\]

\textbf{Distributive:}\quad x \land (y \lor z) \land x = (x \land y \land x) \lor (x \land z \land x)
\[
x \lor (y \land z) \lor x = (x \lor y \lor x) \land (x \lor z \lor x)
\]

\textbf{Cancellative:}\quad x \lor y = x \lor z \quad \text{and} \quad x \land y = x \land z \implies y = z
\[
x \lor z = y \lor z \quad \text{and} \quad x \land z = y \land z \implies x = y
\]
A skew lattice $(S, \land, \lor)$ is called a **left (resp. right) distributive solution** of the Yang-Baxter equation if $(S, r)$ is a set-theoretic solution of the Yang-Baxter equation, where

$$r : S \times S \to S \times S : (x, y) \mapsto (x \land y, y \lor x) \quad (\text{resp.} \ (y \land x, x \lor y)).$$

**Remark:** $(S, r)$ is idempotent
LEFT AND RIGHT DISTRIBUTIVE SOLUTIONS

\{\text{Left cancellative and distributive SL}\}
\quad = \quad \{\text{Left distributive solution}\}

\{\text{Right cancellative and distributive SL}\}
\quad = \quad \{\text{Right distributive solution}\}
LEFT AND RIGHT DISTRIBUTIVE SOLUTIONS

\{\text{Left cancellative and distributive SL}\} = \{\text{Left distributive solution}\}

\{\text{Right cancellative and distributive SL}\} = \{\text{Right distributive solution}\}

**Left cancellative:** \(x \lor y = x \lor z\) and \(x \land y = x \land z\) \(\Rightarrow y = z\)
LEFT AND RIGHT DISTRIBUTIVE SOLUTIONS

\{\text{Left cancellative and distributive SL}\} = \{\text{Left distributive solution}\}

\{\text{Right cancellative and distributive SL}\} = \{\text{Right distributive solution}\}

\text{Left cancellative: } x \lor y = x \lor z \text{ and } x \land y = x \land z \Rightarrow y = z

\text{Right cancellative: } x \lor z = y \lor z \text{ and } x \land z = y \land z \Rightarrow x = y
LEFT AND RIGHT DISTRIBUTIVE SOLUTIONS

\{\text{Left cancellative and distributive SL}\}
= \{\text{Left distributive solution}\}

\{\text{Right cancellative and distributive SL}\}
= \{\text{Right distributive solution}\}

\text{Left cancellative:}\ x \lor y = x \lor z \text{ and } x \land y = x \land z \Rightarrow y = z
\text{Right cancellative:}\ x \lor z = y \lor z \text{ and } x \land z = y \land z \Rightarrow x = y
\text{Distributive:}\ x \land (y \lor z) \land x = (x \land y \land x) \lor (x \land z \land x)
\hspace{1cm} x \lor (y \land z) \lor x = (x \lor y \lor x) \land (x \lor z \lor x)
A skew lattice \((S, \wedge, \vee)\) is called a \textbf{weak distributive solution} of the Yang-Baxter equation if \((S, r)\) is a set-theoretic solution of the Yang-Baxter equation, where

\[
r : S \times S \to S \times S : (x, y) \mapsto (x \wedge y \wedge x, x \vee y \vee x).
\]

\textbf{Remark:} \((S, r)\) is idempotent.
WEAK DISTRIBUTIVE SOLUTIONS

{Simply cancellative, distributive and lower symmetric SL} = 
{Weak distributive solution}
WEAK DISTRIBUTIVE SOLUTIONS

\{\text{Simply cancellative, distributive and lower symmetric SL}\} = \{\text{Weak distributive solution}\}

\textbf{Simply cancellative: } x \lor y \lor x = x \lor z \lor x \text{ and } \\
x \land y \land x = x \land z \land x \Rightarrow y = z
WEAK DISTRIBUTIVE SOLUTIONS

\{\text{Simply cancellative, distributive and lower symmetric SL}\}
\begin{align*}
&= \\
&\{\text{Weak distributive solution}\}
\end{align*}

**Simply cancellative:** \(x \lor y \lor x = x \lor z \lor x\) and \(x \land y \land x = x \land z \land x \Rightarrow y = z\)

**Distributive:** \(x \land (y \lor z) \land x = (x \land y \land x) \lor (x \land z \land x)\)
\(x \lor (y \land z) \lor x = (x \lor y \lor x) \land (x \lor z \lor x)\)
WEAK DISTRIBUTIVE SOLUTIONS

\{\text{Simply cancellative, distributive and lower symmetric SL} \}\ = \ \{\text{Weak distributive solution} \}

\textbf{Simply cancellative: } x \lor y \lor x = x \lor z \lor x \ \text{and}
\begin{align*}
x \land y \land x &= x \land z \land x \ \Rightarrow \ y = z
\end{align*}

\textbf{Distributive: } x \land (y \lor z) \land x = (x \land y \land x) \lor (x \land z \land x)
\begin{align*}
x \lor (y \land z) \lor x &= (x \lor y \lor x) \land (x \lor z \lor x)
\end{align*}

\textbf{Lower symmetric: } x \lor y = y \lor x \ \Rightarrow \ x \land y = y \land x
Proposition

Let \((S, \wedge, \vee)\) be a skew lattice. Then, \((S, r)\) is a set-theoretic solution of the Yang-Baxter equation, where

\[ r : S \times S \rightarrow S \times S : (x, y) \mapsto ((x \wedge y) \vee x, y). \]

Remark: \((S, r)\) is idempotent
Strongly and co-strongly distributive SL

\[ \Downarrow \]

Strong distributive solution

\[ \Downarrow \]

Cancellative, distributive SL
Strongly and co-strongly distributive SL

Strong distributive solution

Cancellative, distributive SL

Left cancellative, distributive SL = Left distributive solution
Strongly and co-strongly distributive SL

↓

Strong distributive solution

↓

Cancellative, distributive SL

Left cancellative, distributive SL = Left distributive solution

Right cancellative, distributive SL = Right distributive solution
OVERVIEW

Strongly and co-strongly distributive SL

Strong distributive solution

Cancellative, distributive SL

Left cancellative, distributive SL

Left distributive solution

Simply cancellative, distributive, lower symmetric SL

Weak distributive solution

Right cancellative, distributive SL

Right distributive solution
Strongly and co-strongly distributive SL
→
Strong distributive solution
→
Cancellative, distributive SL

Left cancellative, distributive SL = Left distributive solution

Right cancellative, distributive SL = Right distributive solution

Simply cancellative, distributive, lower symmetric SL = Weak distributive solution

Skew lattice

Solution \( r(x, y) = ((x \wedge y) \vee x, y) \)
Can we use skew lattices to generalize the notions of braces and cycle sets?

Are all degenerate solutions coming from skew lattices?

What can we say about the structure group associated to solutions obtained from skew lattices?