ON THE STRUCTURE MONOID AND ALGEBRA OF LEFT NON-DEGENERATE SET-THEORETIC SOLUTIONS TO THE YANG-BAXTER EQUATION

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(joint work w. E. Jespers, Ł. Kubat)
YANG-BAXTER AND ALGEBRAIC STRUCTURES

Definition
A set-theoretic solution to the Yang-Baxter equation is a tuple 
\((X, r)\), where \(X\) is a set and \(r : X \times X \rightarrow X \times X\) a function such that (on \(X^3\))

\[
(id_X \times r)(r \times id_X)(id_X \times r) = (r \times id_X)(id_X \times r)(r \times id_X).
\]

For further reference, denote \(r(x, y) = (\lambda_x(y), \rho_y(x))\).
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For further reference, denote $r(x, y) = (\lambda_x(y), \rho_y(x))$.

A set-theoretic solution $(X, r)$ is called

- left (resp. right) non-degenerate, if $\lambda_x$ (resp. $\rho_y$) is bijective,
- non-degenerate, if it is both left and right non-degenerate,
- involutive, if $r^2 = \text{id}_{X \times X}$,
- squarefree, if for any $x \in X$, we have $r(x, x) = (x, x)$. 
APPLIED CALCULUS EQUATION

- Statistical Physics (work of Yang and Baxter),
- Construction of Hopf Algebras,
- Knot theory (Reidemeister III, colourings),
- Quadratic algebras.
Definition

Let \((X, r)\) be a set-theoretic solution of the Yang-Baxter equation. Then the monoid

\[
M(X, r) = \langle \forall x \in X \mid xy = \lambda_x(y)\rho_y(x) \rangle,
\]

is called the structure monoid of \((X, r)\).
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The group \(G(X, r)\) generated by the same presentation is called the structure group of \((X, r)\).
Theorem (ESS, LYZ, S, GV)

Let \((X, r)\) be a non-degenerate solution to YBE, then there exists a unique solution \(r_G\) on the group \(G(X, r)\) such that the associated solution \(r_G\) satisfies

\[ r_G(i \times i) = (i \times i)r, \]

where \(i : X \to G(X, r)\) is the canonical map.
Theorem (ESS, LYZ, S, GV)

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However, there exists a unique solution \(r_M\) on \(M(X, r)\) such that \(r_M|_{X \times X} = r\).
Theorem (GIVdB, JO)

Let \((X, r)\) be a finite, involutive non-degenerate set-theoretic solution. Then, \(G(X, r)\) is a group of \(I\)-type.

In particular, \(G(X, r)\) is a regular subgroup of \(\mathbb{Z}^{\left|X\right|} \rtimes \text{Sym}(X)\) and \(M(X, r)\) is a regular submonoid of \(\mathbb{N}^{\left|X\right|} \rtimes \text{Sym}(X)\).
Definition
Let \((X, r)\) be a set-theoretic solution. Denote the monoid

\[
A(X, r) = \langle x \in X \mid x \lambda_x(y) = \lambda_x(y) \lambda_{\rho_y(x)}(\lambda_x(y)) \rangle.
\]

If \((X, r)\) is left non-degenerate, then for any \(x \in X\) there exists a bijective map \(\sigma_x : X \rightarrow X\) such that

\[
A(X, r) = \langle x \in X \mid xy = y \sigma_y(x) \rangle.
\]

Furthermore, \(s(x, y) = (y, \sigma_y(x))\) defines a non-degenerate set-theoretic solution.
Theorem (LV, JKA)

Let \((X, r)\) be a left non-degenerate set-theoretic solution. Then, \(M(X, r)\) is a regular submonoid of \(A(X, r) \rtimes \text{Sym}(X)\),

where \(x \in X\) is embedded as \((x, \lambda_x)\).
For bijective left non-degenerate set-theoretic solutions, one extends $\sigma : X \rightarrow X$ to an endomorphism $\sigma_a$ on $A(X, r)$, where $a \in A(X, r)$.

**Theorem**

Let $(X, r)$ be a finite bijective left non-degenerate solution. Then, there exists a positive integer $d$ such that $a^d$ is central in $A(X, r)$ for every $a \in A(X, r)$. 
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**Theorem**

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Furthermore, $M(X, r)$ is a central-by-finite monoid
Theorem

Let \((X, r)\) be a finite bijective left non-degenerate solution and \(K\) a field. Then, \(KM = KM(X, r)\) is a Noetherian PI-algebra, with

\[
\text{ClKdim}(KM) = \text{GKdim}(KM) = \text{rk}(M) = \text{rk}(A) \leq |X|.
\]
Conjecture
Let $(X, r)$ be a finite bijective left non-degenerate solution. Does the cancellativity of $M(X, r)$ imply that $(X, r)$ is involutive?

Theorem
Let $(X, r)$ be a finite bijective left non-degenerate solution. Then the following are equivalent:

- $(X, r)$ is an involutive solution,
- $M(X, r)$ is a cancellative monoid,
- $KM$ is a prime algebra,
- $KM$ is a domain,
- $GKdim(KM) = |X|$.
Since every element in $A(X, r)$ is normal, it follows that every prime ideal is determined by invariant subsets of $X$ under certain $\sigma_X$.

**Theorem**

Let $(X, r)$ be a bijective square-free finite left non-degenerate solution. Then every prime ideal $P$ of $M(X, r)$ of height $k$ is determined by prime ideals $Q_1, \ldots, Q_n$ of $A(X, r)$ of height $k$, i.e.

$$P = (Q_1 \cap \ldots \cap Q_n)^e.$$
Can we describe prime ideals of the algebra KM? Let us first consider prime ideals not intersecting the monoid.
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**Theorem**

*Let* $(X, r)$ *be a finite bijective left non-degenerate solution. Then there exists an inclusion preserving bijection between prime ideals of* $KG(X, r)$ *and prime ideals* $P$ *of* $KM$ *with* $P \cap M = \emptyset$. *
Let $Y \subseteq X$. Denote $M_Y = \bigcap_{y \in Y} yM$ and $D_Y = M_Y \setminus \bigcup_{x \in X \setminus Y} M\{x\}$. 
DIVISIBILITY IN $M$

Let $Y \subseteq X$. Denote $M_Y = \bigcap_{y \in Y} yM$ and $D_Y = M_Y \setminus \bigcup_{x \in X \setminus Y} M\{x\}$.

**Theorem**

Let $(X, r)$ be a finite bijective left non-degenerate solution. Let $P$ be a prime ideal in $KM$ with $P \cap M \neq \emptyset$. Then,

$$P \cap M = \bigcup_{Y \in \mathcal{F}} D_Y,$$

where $\mathcal{F} = \{Y \subseteq X \mid D_Y \cap P \neq \emptyset\}$. 
Study $M(X, r)$ and $KM(X, r)$ for idempotent solutions => Talk Ł. Kubat,

Can prime ideals of $M(X, r)$ and $A(X, r)$ be related for non-square-free solutions?

Study $M(X, r)$ and $KM(X, r)$ for general left non-degenerate solutions.
REFERENCES