Global local conjectures and the Bonnafé–Dat–Rouquier Morita equivalence

Lucas Ruhstorfer
Bergische Universität Wuppertal

June 12th, 2019
Motivation: The Alperin-McKay conjecture

Notation:

- $G$ a finite group and $\ell$ a prime with $\ell | |G|$.
- $(K, O, k)$ an $\ell$-modular system large enough.
- $\text{Irr}(G)$ the set of irreducible $K$-characters.
- $B$ be an $\ell$-block of $O_G$ with defect group $D$.
- $b$ the Brauer correspondent of $B$ in $N_{G}(D)$.

Conjecture (Alperin-McKay conjecture): $|\text{Irr}_0(B)| = |\text{Irr}_0(b)|$, where $\text{Irr}_0(B) = \{ \chi \in \text{Irr}(B) | \chi(1) = |G : D| \ell \}$. 

Motivation: The Alperin-McKay conjecture

Notation:
- $G$ a finite group and $\ell$ a prime with $\ell | |G|$. 

Conjecture (Alperin-McKay conjecture)

$|\text{Irr}_0(B)| = |\text{Irr}_0(b)|$, where $\text{Irr}_0(B) = \{\chi \in \text{Irr}(B) | \chi(1) \ell = |G|/|D|\}$. 

Motivation: The Alperin-McKay conjecture

Notation:

- $G$ a finite group and $\ell$ a prime with $\ell | |G|$.
- $(K, O, k)$ an $\ell$-modular system large enough.
Motivation: The Alperin-McKay conjecture

Notation:
- $G$ a finite group and $\ell$ a prime with $\ell \mid |G|$.
- $(K, \mathcal{O}, k)$ an $\ell$-modular system large enough.
- $\text{Irr}(G)$ the set of irreducible $K$-characters.
Motivation: The Alperin-McKay conjecture

Notation:

- $G$ a finite group and $\ell$ a prime with $\ell \mid |G|$.  
- $(K, \mathcal{O}, k)$ an $\ell$-modular system large enough.  
- $\text{Irr}(G)$ the set of irreducible $K$-characters.  
- $B$ be an $\ell$-block of $\mathcal{O}G$ with defect group $D$.  

Conjecture (Alperin-McKay conjecture)

$$|\text{Irr}_0(B)| = |\text{Irr}_0(b)|,$$

where $\text{Irr}_0(B) = \{\chi \in \text{Irr}(B) | \chi(1) = |G : D|\}$.
Motivation: The Alperin-McKay conjecture

Notation:

- $G$ a finite group and $\ell$ a prime with $\ell | |G|$. 
- $(K, O, k)$ an $\ell$-modular system large enough. 
- $\text{Irr}(G)$ the set of irreducible $K$-characters. 
- $B$ be an $\ell$-block of $OG$ with defect group $D$. 
- $b$ the Brauer correspondent of $B$ in $N_G(D)$. 

Conjecture (Alperin-McKay conjecture)

$|\text{Irr}_0(B)| = |\text{Irr}_0(b)|$, where $\text{Irr}_0(B) = \{ \chi \in \text{Irr}(B) | \chi(1) = |G:D| \}$. 

\[ 2 / 8 \]
Motivation: The Alperin-McKay conjecture

Notation:

- $G$ a finite group and $\ell$ a prime with $\ell \mid |G|$.
- $(K, \mathcal{O}, k)$ an $\ell$-modular system large enough.
- $\text{Irr}(G)$ the set of irreducible $K$-characters.
- $B$ be an $\ell$-block of $\mathcal{O}G$ with defect group $D$.
- $b$ the Brauer correspondent of $B$ in $N_G(D)$.

Conjecture (Alperin-McKay conjecture)

$$|\text{Irr}_0(B)| = |\text{Irr}_0(b)|,$$
Motivation: The Alperin-McKay conjecture

Notation:

- $G$ a finite group and $\ell$ a prime with $\ell \mid |G|$. 
- $(K, \mathcal{O}, k)$ an $\ell$-modular system large enough. 
- $\text{Irr}(G)$ the set of irreducible $K$-characters. 
- $B$ be an $\ell$-block of $\mathcal{O}G$ with defect group $D$. 
- $b$ the Brauer correspondent of $B$ in $N_G(D)$. 

Conjecture (Alperin-McKay conjecture)

$$|\text{Irr}_0(B)| = |\text{Irr}_0(b)|,$$
where $\text{Irr}_0(B) = \{\chi \in \text{Irr}(B) \mid \chi(1)_\ell = |G : D|_\ell\}$. 
The reduction theorem

Theorem (Sp"ath '13)

The Alperin-McKay conjecture holds for all groups and primes if the so-called inductive Alperin-McKay ($iAM$) condition holds for all blocks of quasi-simple groups.

The $iAM$-condition holds for an $\ell$-block $B$ of a quasi-simple group $G$ if there exists an $\text{Aut}(G B, D)$-equivariant bijection $\Omega : \text{Irr}_0(B) \to \text{Irr}_0(b)$, preserving Clifford theory with respect to $G \rtimes \text{Aut}(G B, D)$. 
The Alperin-McKay conjecture holds for all groups and primes if the so-called inductive Alperin-McKay (iAM) condition holds for all blocks of quasi-simple groups.
The reduction theorem

Theorem (Späth '13)

The Alperin-McKay conjecture holds for all groups and primes if the so-called inductive Alperin-McKay (\textbf{iAM}) condition holds for all blocks of quasi-simple groups.

The $\textbf{iAM}$-condition holds for an $\ell$-block $B$ of a quasi-simple group $G$ if there exists an $\text{Aut}(G)_B, D$-equivariant bijection

$$\Omega : \text{Irr}_0(B) \rightarrow \text{Irr}_0(b),$$
The reduction theorem

Theorem (Späth ’13)

The Alperin-McKay conjecture holds for all groups and primes if the so-called inductive Alperin-McKay (iAM) condition holds for all blocks of quasi-simple groups.

The iAM-condition holds for an \( \ell \)-block \( B \) of a quasi-simple group \( G \) if there exists an \( \text{Aut}(G)_B, D \)-equivariant bijection

\[
\Omega : \text{Irr}_0(B) \to \text{Irr}_0(b),
\]

preserving Clifford theory with respect to \( G \triangleleft G \rtimes \text{Aut}(G)_B, D \).
Let $G$ be a connected reductive group with Frobenius $F: G \to G$ defining an $F_\mathbb{Q}$-structure, $\ell \nmid q$.

Let $(G^*, F^*)$ denote the dual group of $(G, F)$.

**Theorem (Broué-Michel '89)**

We have a decomposition:

$$O_{G^*} F^* G^* \cong \bigoplus (s) O_{G^*} F^* (s)^{-1} \mod$$

where $(s)$ runs over the set of $(G^*)^* F^*$-conjugacy classes of semisimple elements of $(G^*)^*$ of $\ell'$-order.

**Aim:** Understand the Representation theory of $O_{G^*} F^* G^* \mod$ for a fixed series $(s)$. 


Let $G$ be a connected reductive group with Frobenius $F : G \to G$ defining an $\mathbb{F}_q$-structure, $\ell \nmid q$. 

Theorem (Broué-Michel '89) We have a decomposition:

$$O_{G^F} \sim \bigoplus (s) O_{G^F s^{-1}}$$

where $(s)$ runs over the set of $(G^*)^F$-conjugacy classes of semisimple elements of $(G^*)^F$ of $\ell'$-order.

Aim: Understand the representation theory of $O_{G^F s^{-1}}$ for a fixed series $(s)$. 


Let $G$ be a connected reductive group with Frobenius $F : G \to G$ defining an $\mathbb{F}_q$-structure, $\ell \nmid q$.
Let $(G^*, F^*)$ denote the dual group of $(G, F)$.

**Theorem (Broué-Michel ’89)**

We have a decomposition:

$$ \mathcal{O}G^F - \text{mod} \cong \bigoplus_{(s)} \mathcal{O}G^F e_s^{G^F} - \text{mod} $$

where $(s)$ runs over the set of $(G^*)^{F^*}$-conjugacy classes of semisimple elements of $(G^*)^{F^*}$ of $\ell'$-order.
Let $G$ be a connected reductive group with Frobenius $F : G \to G$ defining an $\mathbb{F}_q$-structure, $\ell \nmid q$.
Let $(G^*, F^*)$ denote the dual group of $(G, F)$.

**Theorem (Broué-Michel '89)**

We have a decomposition:

$$\mathcal{O}_G^F \mod \cong \bigoplus_{(s)} \mathcal{O}_{G^*}^{F^*} e_s^{G^*} \mod (s)$$

where $(s)$ runs over the set of $(G^*)^{F^*}$-conjugacy classes of semisimple elements of $(G^*)^{F^*}$ of $\ell'$-order.

**Aim:** Understand the Representation theory of $\mathcal{O}_G^F e_s^{G^*} \mod$ for a fixed series $(s)$. 

Let $L$ Levi subgroup of $G$ with $F(L) = L$ and Levi decomposition $P = L \rtimes U$. 
Let $L$ Levi subgroup of $G$ with $F(L) = L$ and Levi decomposition $P = L \ltimes U$. Consider the Deligne–Lusztig variety

$$Y^G_U = \{ gU \in G/U \mid g^{-1}F(g) \in UF(U) \}$$

with $G^F \times (L^F)^\text{opp}$-action.
Jordan decomposition for characters

Let $L$ Levi subgroup of $G$ with $F(L) = L$ and Levi decomposition $P = L \ltimes U$. Consider the Deligne–Lusztig variety

$$Y^G_U = \{gU \in G/U \mid g^{-1}F(g) \in UF(U)\}$$

with $G^F \times (L^F)^\text{opp}$-action.

**Theorem (Bonnafé–Rouquier ’03)**

Let $C_{G^*}(s) \subseteq L^*$. Then the bimodule $H_{c}^{\dim(Y^G_U)}(Y^G_U, \mathcal{O})e_s^{L^F}$ induces a Morita equivalence between $\mathcal{O}G^F e_s^{L^F} - \text{mod}$ and $\mathcal{O}L^F e_s^{G^F} - \text{mod}$. 

- Reduces questions about blocks to a question about quasi-isolated blocks of Levi subgroups.
- $O_L e_s^{G^F} - \text{mod}$ and $O_G e_s^{L^F} - \text{mod}$ splendid Rickard equivalence $\Rightarrow$ Rickard equivalences on the level of local subgroups (Bonnafé–Dat–Rouquier ’17).
Jordan decomposition for characters

Let $L$ Levi subgroup of $G$ with $F(L) = L$ and Levi decomposition $P = L \rtimes U$. Consider the Deligne–Lusztig variety

$$Y^G_U = \{ gU \in G/U \mid g^{-1}F(g) \in UF(U) \}$$

with $G^F \times (L^F)^{opp}$-action.

Theorem (Bonnafé–Rouquier ’03)

Let $C_{G^*}(s) \subseteq L^*$. Then the bimodule $H^\text{dim}(Y^G_U)(Y^G_U, \mathcal{O})e_s^{L^F}$ induces a Morita equivalence between $\mathcal{O}G^F e_s^{L^F} - \text{mod}$ and $\mathcal{O}L^F e_s^{G^F} - \text{mod}$.

- Reduces questions about blocks to a question about quasi-isolated blocks of Levi subgroups.
Jordan decomposition for characters

Let $L$ Levi subgroup of $G$ with $F(L) = L$ and Levi decomposition $P = L \ltimes U$. Consider the Deligne–Lusztig variety

$$Y^G_U = \{ gU \in G/U \mid g^{-1}F(g) \in UF(U) \}$$

with $G^F \times (L^F)^{\text{opp}}$-action.

**Theorem (Bonnafé–Rouquier ’03)**

Let $C_{G^*}(s) \subseteq L^*$. Then the bimodule $H_c^{\dim(Y^G_U)}(Y^G_U, \mathcal{O})e^L_F$ induces a Morita equivalence between $\mathcal{O}G^F e^L_F - \text{mod}$ and $\mathcal{O}L^F e^G_F - \text{mod}$.

- Reduces questions about blocks to a question about quasi-isolated blocks of Levi subgroups.
- $\mathcal{O}L^F e^L_F$ and $\mathcal{O}G^F e^G_F$ splendid Rickard equivalence $\Rightarrow$ Rickard equivalences on the level of local subgroups (Bonnafé–Dat–Rouquier ’17).
We want to apply the Bonnafé–Dat–Rouquier reduction to the iAM-condition.
We want to apply the Bonnafé–Dat–Rouquier reduction to the iAM-condition.

\[
\begin{align*}
\text{Bl}(L^F) & \ni C \quad \longrightarrow \quad B \in \text{Bl}(G^F) \\
\text{Bl}(N_{L^F}(D)) & \ni c \quad \longrightarrow \quad b \in \text{Bl}(N_{G^F}(D))
\end{align*}
\]
We want to apply the Bonnafé–Dat–Rouquier reduction to the \textbf{iAM}-condition.

\[
\begin{array}{ccc}
\text{Bl}(L^F) \ni C & \longrightarrow & B \in \text{Bl}(G^F) \\
\downarrow & & \downarrow \\
\text{Bl}(N_{L^F}(D)) \ni c & \longrightarrow & b \in \text{Bl}(N_{G^F}(D))
\end{array}
\]

Two main tasks:
We want to apply the Bonnafé–Dat–Rouquier reduction to the \textbf{iAM}-condition.

\[
\begin{align*}
\text{Bl}(L^F) & \ni C \quad \rightarrow \quad B \in \text{Bl}(G^F) \\
\text{Bl}(N_{L^F}(D)) & \ni c \quad \rightarrow \quad b \in \text{Bl}(N_{G^F}(D))
\end{align*}
\]

Two main tasks:

- Lift to Morita equivalence to include automorphisms coming from \(E = \langle \gamma, F_0 \rangle\) similar as in Julian’s talk.
We want to apply the Bonnafé–Dat–Rouquier reduction to the iAM-condition.

\[
\begin{align*}
B \ell(L^F) & \ni C \quad \rightarrow \quad B \in B \ell(G^F) \\
B \ell(N_{L^F}(D)) & \ni c \quad \rightarrow \quad b \in B \ell(N_{G^F}(D))
\end{align*}
\]

Two main tasks:
- Lift to Morita equivalence to include automorphisms coming from \( E = \langle \gamma, F_0 \rangle \) similar as in Julian’s talk.
- Find a local equivalence on the level of normalizers satisfying similar properties.
Main results

Theorem (R. ’19)

Suppose that the order of $\gamma$ is coprime to $\ell$. Then $\mathcal{O}_L^F E e_s^{L^F} \mod$ and $\mathcal{O}_G^F E e_s^{G^F} \mod$ are Morita equivalent.
Main results

Theorem (R. ’19)

Suppose that the order of $\gamma$ is coprime to $\ell$. Then $O_{L^F E} e_s^{L^F} – \text{mod}$ and $O_{G^F E} e_s^{G^F} – \text{mod}$ are Morita equivalent.

Proof: The module $H^c_{\text{dim}(Y_U^G)}(Y_U^G, O)e_s^{L^F}$ extends to an $O_{G^F} \times (L^F)^{\text{opp}} \Delta(E)$-module $M$. 
Main results

**Theorem (R. ’19)**

Suppose that the order of $\gamma$ is coprime to $\ell$. Then $\mathcal{O}L^F E e_s^{L^F} \mod$ and $\mathcal{O}G^F E e_s^{G^F} \mod$ are Morita equivalent.

**Proof:** The module $H^c_{\dim(Y_U^G)}(Y^G_U, \mathcal{O})e_s^{L^F}$ extends to an $\mathcal{O}G^F \times (L^F)^{\text{opp}} \Delta(E)$-module $M$. Then $\text{Ind}_{G^F \times (L^F)^{\text{opp}} \Delta(E)}^{G^F \times (L^F)^{\text{opp}} \Delta(E)}(M)$ induces a Morita equivalence between $\mathcal{O}L^F E e_s^{L^F} \mod$ and $\mathcal{O}G^F E e_s^{G^F} \mod$ (Marcus ’96).
Main results

**Theorem (R. ’19)**

Suppose that the order of $\gamma$ is coprime to $\ell$. Then

$$\mathcal{O} L^F E e^L_s - \text{mod} \text{ and } \mathcal{O} G^F E e^G_s - \text{mod}$$

are Morita equivalent.

**Proof:** The module $H_c^{\dim(Y_U^G)}(Y_U^G, \mathcal{O}) e^L_s$ extends to an $\mathcal{O} G^F \times (L^F)^{\text{opp}} \Delta(E)$-module $M$. Then $\text{Ind}_{G^F \times (L^F)^{\text{opp}} \Delta(E)}^{G^F \times (L^F)^{\text{opp}} \Delta(E)}(M)$ induces a Morita equivalence between $\mathcal{O} L^F E e^L_s - \text{mod}$ and $\mathcal{O} G^F E e^G_s - \text{mod}$ (Marcus ’96).

**Consequences**

We have similar equivalence on the level of normalizers
Theorem (R. ’19)

Suppose that the order of $\gamma$ is coprime to $\ell$. Then $O_{L^F E} e_{s}^{L^F} \mod$ and $O_{G^F E} e_{s}^{G^F} \mod$ are Morita equivalent.

Proof: The module $H_{c}^{\dim(Y_{U}^{G})}(Y_{U}^{G}, O) e_{s}^{L^F}$ extends to an $O_{G^F} \times (L^F)^{\text{opp}} \Delta(E)$-module $M$. Then $\text{Ind}_{G^F \times (L^F)^{\text{opp}} \Delta(E)}^{G^F E \times (L^F E)^{\text{opp}}} (M)$ induces a Morita equivalence between $O_{L^F E} e_{s}^{L^F} \mod$ and $O_{G^F E} e_{s}^{G^F} \mod$ (Marcus ’96).

Consequences

We have similar equivalence on the level of normalizers

⇒ Reduce the verification of the iAM-conditions to quasi-isolated blocks of Levi subgroups (Work in progress).
Thank you!