Groups with few $p'$-character degrees
Joint work with E. Giannelli and A. Schaeffer Fry

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Groups, Rings and Associated Structures 2019
Introduction: the set up

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- $\text{Irr}_{p'}(G) = \{\chi \in \text{Irr}(G) \mid p \nmid \chi(1)\}$
- $\text{cd}_{p'}(G) = \{\chi(1) \in \text{cd}(G) \mid p \nmid \chi(1)\}$
The Itô-Michler theorem

\[ \text{cd}_{p'}(G) = \text{cd}(G). \]
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- \( \text{cd}_{p'}(G) = \text{cd}(G) \).
- Using standard character theory: if \( G \) has normal and abelian Sylow \( p \)-subgroup, then \( p \nmid \chi(1) \) for all \( \chi \in \text{Irr}(G) \) \( \Rightarrow \) \( \text{cd}_{p'}(G) = \text{cd}(G) \).
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**Theorem (Itô-Michler (1951, 1986))**

$\text{cd}_{p'}(G) = \text{cd}(G)$ if and only if $G$ has a normal and abelian Sylow $p$-subgroup.

- One of the first applications of the CFSG to character theory.
An extension of the Itô-Michler theorem

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- Next minimal situation: \( \text{cd}(G) = \text{cd}_{p'}(G) \cup \{a\} \).

Theorem (M. Isaacs, A. Moretó, G. Navarro and P. H. Tiep, 2009)

Let \( p \) be a prime, let \( G \) be a finite group, and let \( P \in \text{Syl}_p(G) \). If \( \text{cd}(G) = \text{cd}_{p'}(G) \cup \{a\} \), then \( P \) is metabelian.

No knowledge of how normal is \( P \).


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- The dual situation:
  \[ \text{cd}_{p'}(G) = \{1\}. \]
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**Theorem (Thompson (1970))**

*If $\text{cd}_{p'}(G) = \{1\}$, then $G$ has a normal $p$-complement.*
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**Theorem (Thompson (1970))**

If \( cd_{p'}(G) = \{1\} \), then \( G \) has a normal \( p \)-complement.

**Theorem (Berkovich (1995))**

If \( cd_{p'}(G) = \{1\} \), then \( G \) is solvable.
Some extensions of Thompson’s theorem

- Taking into account two primes.

**Theorem (E. Giannelli, M. Schaeffer Fry, C. Vallejo, ’19))**

Let \( \pi = \{p, q\} \) a set of primes. If \( \text{Irr}_{\pi'}(G) = \{1_G\} \) then \( G = 1 \).
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**Theorem (E. Giannelli, R., M. Schaeffer Fry, ’19)**

Let $p > 3$ be a prime. If $\text{cd}_{p'}(G) = \{1, m\}$, then $G$ is solvable and $O^{pp'}p'(G) = 1$. 
On our theorem

- Many counterexamples for $p = 2$, some for $p = 3$. 

Theorem (E. Giannelli, R., M. Schaeffer Fry, '19)

Let $S$ be a non-abelian simple group and let $p > 3$ be a prime. Then there exist non-linear $\alpha, \beta \in \text{Irr}_p(S)$ with:

1. $\alpha$ extends to $\text{Aut}(S)$.
2. $\beta$ is $P$-invariant for all $P \leq \text{Aut}(S)$, $p$-subgroup.
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\begin{align*}
\chi &\in \text{Irr}(G) \\
\chi(1) &= \theta(1) = \alpha(1)' = p' \\
\Rightarrow \chi(1) &= m \\
\theta &= \alpha \times \alpha \times \cdots \times \alpha \in \text{Irr}(N) \\
N &= S \times S \times \cdots \times S
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- $N$ is abelian and $G$ solvable.
THANK YOU!