Perpetuants: a lost treasure

Claudio Procesi (Joint work with H. Kraft)

Claudio Procesi Roma, La Sapienza

Spa, 15–June–2019
Introduction

Invariant Theory

Binary forms

U–invariants
Perpetuants

The theorem of Stroh
The potenziant

A sketch of the proof
is one of the several concepts invented (in 1882) by J. J. Sylvester in his investigations of *covariants* for binary forms.
It appears in one of the first issues of the American Journal of Mathematics which he had founded a few years before.

*Perpetuant* is a concept of *Invariant theory*, and a name which will hardly appear in a mathematical paper of the last 70 years, (and sometimes apperas with a wrong formulation).

This is due to the complex history of invariant theory which was at some time declared dead only to resurrect several decades later.
I learned of this word from Gian-Carlo Rota who pronounced it with an enigmatic smile.
Invariant Theory
Algebraic invariant theory in its simplest form

treats of a group $G$ of linear transformations on a vector space $V$.

- The action extends to polynomial functions $S[V^*]$ on $V$, by the formula

$$ (g \cdot f)(v) := f(g^{-1}v) $$

- A polynomial $f(v)$ is $G$ invariant if

$$ (g \cdot f)(v) := f(g^{-1}v) = f(v), \quad \forall g \in G, \ v \in V. $$
Algebraic invariant theory in its simplest form

the invariants form a subalgebra of $S[V^*]$ denoted by $S[V^*]^G$.

In 19th century the theory was developed essentially for $V$ the space of homogeneous polynomials of some degree $q$ in $k$–variables (called *forms* or *k–ary quantics*)

and $G$ the special linear group $SL(k, \mathbb{C})$ acting on these variables or for the direct sum of copies of such forms.
Algebraic invariant theory in its simplest form

For $k = 2$ one then speaks of *binary quantics* or $q$–antics:

$$f(x, y) = \sum_{i=0}^{q} \binom{q}{i} a_i x^{q-i} y^i$$

a general binary quantic.

$q = 2, 3, 4, 5, \cdots$ binary quadratic, cubic, quartic, quintic, etc. The group acting is $SL(2, \mathbb{C})$ (acts on $x, y$).

- This is a $q + 1$ dimensional vector space $V_q$
- The polynomial functions over $V_q$ are $\mathbb{C}[a_0, a_1, a_2, \ldots, a_q]$.  

The binomial coefficients are introduced to simplify some of the formulas.
Introduction

binary forms

Perpetuants

The theorem of Stroh

The potenziante

A sketch of the proof

Back to 1850

*Binary forms*
The invariants of a general binary quantic are thus special polynomials in the variables $a_0, a_1, \ldots, a_q$. The space of these polynomials can be bigraded by defining the weight $g$ of $a_i$ to be $i$ so that

$$g\left(\prod_{j=0}^{q} a_j^{h_j}\right) = \sum_{j=1}^{q} j \cdot h_j.$$ 

A polynomial with terms all of the same weight is called *isobaric*. 

**A quick course on binary forms**
A quick course on binary forms

One easily sees that, for a homogeneous invariant of a binary $q$–form there is a relation between

the degree $k$ and the weight $g$

$$g = \frac{q \cdot k}{2}.$$

As example the discriminant of the cubic ($q = 3$):

$$D = 3a_1^2a_2^2 + 6a_0a_1a_2a_3 - 4a_1^3a_3 - 4a_0a_2^3 - a_0^2a_3^2$$

of degree 4 and weight $6 = \frac{3 \cdot 4}{2}$, generates the algebra of invariants of the cubic. For $q = 2$ we have also the discriminant $a_1^2 - 2a_0a_2$. 
A quick course on binary forms

Classically generators for invariants of binary $q$–forms were known only for forms of degree $q = 2, 3, 4, 5, 6$ and $8$ (partial results by Von Gall for degree $7$).

Now with the help of computers a few other cases have been analyzed for $q \leq 12$. 
semi-invariants or $U$-invariants
A quick course on binary forms semi–invariants

the theory classically is developed using the covariants. These can be described through the semi–invariants or in modern terms \( U \)–invariants, that is the invariants under the subgroup of \( SL(2, \mathbb{C}) \)

\[
U := \left\{ \begin{vmatrix} 1 & \lambda \\ 0 & 1 \end{vmatrix}, \quad \lambda \in \mathbb{C} \right\}
\]

acting as

\[
\begin{vmatrix} 1 & \lambda \\ 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} x + \lambda y \\ y \end{vmatrix}.
\]
It is equivalent to set $y = 1$ and replace binary forms by *polynomials* in $x$

$$p(x) = \sum_{i=0}^{q} \binom{q}{i} a_i x^{q-i}$$

a general polynomial of degree $q$.

The action is just transaltion $x \mapsto x - \lambda$. 

**semi–invariants**
denote the ring of $U$–invariants of binary forms of degree $q$ (or polynomials) by

$$S(q) = \mathbb{C}[a_0, a_1, a_2, \ldots, a_q]^U.$$ 

- Then $S(q) = \bigoplus S(q)_{k,g}$ is bigraded, that is it decomposes into a direct sum of components, homogeneous of degree $k$ and isobaric of weight $g$.
- The ring of invariants under $SL(2, \mathbb{C})$ is the subring of $S(q)$ direct sum of the homogeneous and isobaric components with $g = \frac{q \cdot k}{2}$. 
It is useful to use the formalism of *divided powers*

\[ \chi[h] := \frac{\chi^h}{h!} \]

for which the binomial theorem becomes

\[ (\lambda + \mu)[h] = \sum_{i=0}^{h} \lambda^{[h-i]} \mu[i]. \]

---

**Change notations for the \( a_i \) in polynomials:**

\[
f(x) = \sum_{i=0}^{q} a_i x^{[q-i]}, \quad \text{e.g.} \quad a_0 \frac{x^2}{2} + a_1 x + a_2.
\]
of 19th century invariant theory was to

1. describe a \textbf{minimal set of generators} for invariants or for $U$–invariants,
2. and possibly also a minimal set of relations.

Classically generators for $U$–invariants were computed for $q \leq 6$. Already for $q = 5$ we have 23 generators and the computations are quite lengthy.
The generator of degree 18 has millions of terms!
Example of the cubic (new notations for $a_i$)

The algebra of semi–invariants, for the cubic, is generated by 4 elements. The discriminant $D$, and $a_0, H, T$:

$$D = 9a_0^2a_3^2 - 18a_0a_1a_2a_3 + 8a_0a_2^3 + 6a_1^3a_3 - 3a_1^2a_2^2$$

$$a_0, \quad H = a_1^2 - 2a_0a_2, \quad T = a_1^3 - 3a_0a_1a_2 + 3a_0^2a_3$$

the element $a_0$ (degree 1, weight 0), and $H$ of degree 2 and weight 2, $T$ of degree 3 and weight 3.

They are related by the syzygy $H^3 + Da_0^2 - T^2 = 0$ of degree 6 and weight 6.
Example of the quartic

The algebra of semi–invariants, for the quartic is generated by 5 elements $a_0, B, C, H, T$.

\begin{align*}
a_0, \quad H &= a_1^2 - 2a_0 a_2, \quad T = a_1^3 - 3a_0 a_1 a_2 + 3a_0^2 a_3 \\
B &= 2a_0 a_4 - 2a_1 a_3 + a_2^2, \\
C &= 2a_2^3 - 6a_1 a_2 a_3 + 9a_0 a_3^2 + 6a_1^2 a_4 - 12a_0 a_2 a_4.
\end{align*}

Relation \[ 3a_0^2 HB - a_0^3 C - H^3 + T^2 = 0. \]

Notice that now \[ D = -3HB + a_0 C. \]
in modern terms given a graded commutative algebra $A = \bigoplus_{i=0}^{\infty} A_i$ over a field $F = A_0$, setting $I = \bigoplus_{i=1}^{\infty} A_i$ we have that the elements of $I^2$ are decomposable and a minimal set of generators of $A$ is a set of homogeneous elements giving a basis of $I/I^2$.

$I/I^2$ is a graded vector space and it is finite dimensional if and only if $A$ is finitely generated over $F$.

Apply this to $A = S(q)$ and denote by $I_q$ the subspace of $U$–invariants with no constant term.

Then $I_q/I_q^2$ is bigraded by degree and weight and a basic problem is to prove that it is finite dimensional and compute the dimension of its bigraded pieces.
in modern terms given a graded commutative algebra $A = \bigoplus_{i=0}^{\infty} A_i$ over a field $F = A_0$, setting $I = \bigoplus_{i=1}^{\infty} A_i$ we have that the elements of $I^2$ are decomposable and a minimal set of generators of $A$ is a set of homogeneous elements giving a basis of $I/I^2$.

$I/I^2$ is a graded vector space and it is finite dimensional if and only if $A$ is finitely generated over $F$.

Apply this to $A = S(q)$

and denote by $I_q$ the subspace of $U$–invariants with no constant term.

Then $I_q/I_q^2$ is bigraded by degree and weight and a basic problem is to prove that it is finite dimensional and compute the dimension of its bigraded pieces.
Introduction

binary forms
Perpetuants
The theorem of Stroh
The potenzianten
A sketch of the proof

a basic problem

A lot of work by

Cayley, Sylvester, Clebsch, Gordan and others was devoted to this problem with success for forms of degree $q \leq 6$ and partial results in degrees 7,8.

Recent work, with the help of computers, gives results up to degree 12.

A crowning point of this research was Gordan’s proof that these algebras of invariants are finitely generated, but no explicit formulas for either the generators or even for just the weight and degree of these generators is known for $q > 12$. 
The theory was revolutionised by Hilbert at the end of the century. He proved the finiteness theorem of forms in any number of variables and asked, in his 14th problem, if the finiteness theorem is true for every group. Negative answer by Nagata 1958.
We can finally define:

Perpetuants
Stable semi–invariants

The action of $\lambda$ on the coefficients is

$$f(x + \lambda) = \sum_{j=0}^{q} a_j (x + \lambda)^{[q-j]} = \sum_{j=0}^{q} a_j \sum_{i} x^{[q-j-i]} \lambda^i$$

set $k = j + i$, 

$$= \sum_k (\sum_j a_j \lambda^{[k-j]}) x^{[q-k]}.$$ 

$$\lambda \cdot a_k = \sum_{j+h=k} a_j \lambda^{[h]} = \sum_{j=0}^{k} a_j \lambda^{[k-j]},$$

for instance

$$\lambda \cdot a_0 = a_0, \quad \lambda \cdot a_1 = a_0 \lambda + a_1, \quad \lambda \cdot a_2 = a_0 \lambda^{[2]} + a_1 \lambda + a_2,$$

$$\lambda \cdot a_3 = a_0 \lambda^{[3]} + a_1 \lambda^{[2]} + a_2 \lambda + a_3,$$

$$\lambda \cdot a_4 = a_0 \lambda^{[4]} + a_1 \lambda^{[3]} + a_2 \lambda^{[2]} + a_3 \lambda + a_4, \; \ldots.$$
From the formulas it is clear that the ring of $U$–invariants $S(q)$ is contained in the ring of $U$–invariants $S(q + 1)$ and so on.

So one can define the ring of $U$–invariants $S = \bigcup_q S(q)$ as the subring of the polynomial ring:

$$\mathbb{C}[a_0, a_1, a_2, \ldots, a_n, \ldots], \quad a_i, \ i = 0, \ldots, \infty$$

in the infinitely many variables $a_i, \ i = 0, \ldots, \infty$ invariant under the limit action of the 1–parameter subgroup $U$.

This 1–parameter subgroup $\lambda \cdot a_k = \sum_{j=0}^{k} a_j \lambda^{[k-j]}$,

has as infinitesimal generator the differential operator

$$D = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i}, \quad D(a_i) = a_{i-1}, \ D(a_0) = 0.$$
a basic theorem

**Theorem**

The algebra $S = \mathbb{C}[a_0, a_1, a_2, \ldots]^U$ of $U$-invariants is formed by the polynomials $f$ in the infinitely many variables $a_0, a_1, a_2, \ldots$, satisfying $Df = 0$ i.e.:

$$\sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} f(a_0, a_1, a_2, \ldots) = 0, \quad D = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i}.$$
It was quickly discovered that

- an element of $S(q)$, which is indecomposable in $S(q)$, need not remain indecomposable in $S(q + 1)$.

- In other words: the maps $I_q/I_q^2 \to I_{q+1}/I_{q+1}^2$ need not be injective,

- or also: a minimal set of generators for $S(q)$ cannot be completed to one for $S(q + 1)$.

As an example the generator $D$ for $S(3)$ is decomposable in $S(4)$.

$$D = -3HB + a_0 C.$$
**Perpetuants**

**Definition**

A *perpetuant* is an indecomposable element of $S(q)$ which remains indecomposable in all $S(k), \ k \geq q$ hence in $S = \bigcup S(k)$.

In other words it gives an element of $I_q/I_q^2$ which *lives forever*, that is it remains nonzero in all $I_k/I_k^2, \ \forall \ k \geq q$.

In this sense it is *perpetuant*.

In other words, denoting by $I \subset S$ the ideal of positive elements of $S$ *perpetuants* are essentially the elements of $I \setminus I^2$. 
Perpetuants

Thus to describe perpetuants is strictly related to describe minimal sets of generators for the graded algebra \( S = \mathbb{C}[a_0, a_1, a_2, \ldots]U. \)

In other words, denoting by \( I \subset S \) the ideal of positive elements of \( S \) we want to describe \( I/I^2 \).

This space decomposes into a direct sum

\[
I/I^2 = \bigoplus_{n,g \in \mathbb{N}} P_{n,g}
\]

with \( P_{n,g} \) the image of the elements in \( I \) of degree \( n \) and weight \( g \).
Perpetuants

We may define a space of perpetuants as a bigraded subspace

\[ \Pi = \bigoplus_{i,g} \Pi_{i,g} \subset I \]

which maps isomorphically to

\[ I/I^2 = \bigoplus_{n,g \in \mathbb{N}} P_{n,g}. \]

A basis of a space of perpetuants is thus a minimal set of generators for the algebra \( S \) of \( U \) invariants.

In our paper we exhibit a space of perpetuants, our main theorem, see page 57.
The theorem of Emile Stroh
conjectured by MacMahon and proved by Stroh

\[
\sum_{g=0}^{\infty} \dim(P_{n,g}) x^g = \begin{cases} 
\frac{x^{2^{n-1}} - 1}{(1 - x^2)(1 - x^3) \cdots (1 - x^n)} & \text{for } n > 2, \\
\frac{x^2}{(1 - x^2)} & \text{for } n = 2, \\
1 & \text{for } n = 1.
\end{cases}
\]

For \( n \leq 3 \) proved by Sylvester.
The partition function

Notice that in the series

\[
\sum_{i=0}^{\infty} p_i(n)x^i = \frac{1}{(1-x^2)(1-x^3)\cdots(1-x^n)}
\]

the integer \( p_i(n) \) counts the number of ways in which the integer \( i \) can be written as a sum of integers \( 2, 3, \ldots, n \) hard to compute!

In the movie "The Man Who Knew Infinity" there is a competition between Mac Mahon and Srinivasa Ramanujan, to compute \( p_{100}(100) \)!
Introduction

binary forms

Perpetuants

The theorem of Stroh

The potenzianten

A sketch of the proof

MacMahon and Srinivasa Ramanujan
We define a linear map \( E \) from the space of all polynomials in auxiliary variables \( \alpha_1, \ldots, \alpha_n \) (the umbrae) to the space of polynomials of degree \( n \) in the variables \( a_0, a_1, a_2, \ldots \),

\[
E : \mathbb{C}[\alpha_1, \ldots, \alpha_n] \rightarrow \mathbb{C}[a_0, a_1, a_2, \ldots], \quad \alpha_1^{[r_1]} \cdots \alpha_n^{[r_n]} \mapsto a_{r_1} \cdots a_{r_n},
\]

1. a homogeneous polynomial of degree \( g \), in the \( \alpha_i, \; i = 1, \ldots, n \), is mapped to an isobaric polynomial of weight \( g \) in the \( a_j \), and homogeneous of degree \( n \).
2. The map \( E \) commutes with the permutation action on the \( \alpha_i, \; i = 1, \ldots, n \)

\[
E(\alpha_1^{[3]} \alpha_2^{[2]}) = E(\alpha_3^{[3]} \alpha_1^{[2]}) = a_0^{n-2} a_2 a_3 \quad \text{and} \quad E(\alpha_i^{[2]} \alpha_j^{[2]}) = a_0^{n-2} a_2^2.
\]
The map $E$ is not a homomorphism but if $f(\alpha_1, \ldots, \alpha_h), \ g(\alpha_{h+1}, \ldots, \alpha_n)$ are in disjoint variables we have

$$E(f(\alpha_1, \ldots, \alpha_h) \ g(\alpha_{h+1}, \ldots, \alpha_n))$$

$$= E(f(\alpha_1, \ldots, \alpha_h))E(g(\alpha_{h+1}, \ldots, \alpha_n)).$$
The basic formula of umbral calculus

\[ E \circ \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_i} = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} \circ E = D \circ E. \]
The potenzante
The potenziante

Stroh defines the potenziante $\pi_{n,g} = \pi_{n,g}(\lambda; a)$ by

\[
\pi_{n,g} := E \left( \left( \sum_{j=1}^{n} \lambda_j \alpha_j \right)[g] \right) = \sum_{\substack{r_1, \ldots, r_n \in \mathbb{N} \\ r_1 + \cdots + r_n = g}} \lambda_1^{r_1} \cdots \lambda_n^{r_n} a_{r_1} \cdots a_{r_n}
\]

where the $\alpha_1 \ldots, \alpha_n$ are all umbrae and $\lambda$ some dual variables.

We now use the symmetry of the umbrae
The potenziantes

**Definition**

Denote by $\Sigma_{n,g} \subset \mathbb{C}[\lambda_1, \ldots, \lambda_n]$ the subspace of *symmetric polynomials* in $\lambda_1, \ldots, \lambda_n$ which are homogeneous of degree $g$.

This space has different combinatorial bases all indexed by partitions of $g$ into $n$ parts.

Many are in fact bases for $\mathbb{Z}[\lambda_1, \ldots, \lambda_n]^{S_n}$. Where $S_n$ is the symmetric group permuting the $\lambda_i$. 
We first take as basis of $\Sigma_{n,g}$ the simplest:

*total monomial sums* $m_{h_1,\ldots,h_n}$

i.e. the sum over the $S_n$-orbit of $\lambda_1^{h_1} \cdots \lambda_n^{h_n}$ where $h_1 \geq h_2 \geq \cdots \geq h_n \geq 0$ and $h_1 + \cdots + h_n = g$:

$$m_{h_1,\ldots,h_n}(\lambda) := \sum_{S_n\text{-orbit}} \lambda_1^{h_1}_{\sigma(1)} \cdots \lambda_n^{h_n}_{\sigma(n)}.$$
In $\pi_{n,g}(\lambda; a)$ the total monomial sum $m_{h_1,\ldots,h_n}(\lambda)$ has as coefficient the product $a_{h_1}a_{h_2}\cdots a_{h_n}$:

$$
\pi_{n,g}(\lambda; a) = E\left(\left(\sum_{r=1}^{n} \lambda_r \alpha_r \right)[g]\right) = \sum_{\substack{h_1 \geq \cdots \geq h_n \geq 0 \\ h_1 + \cdots + h_n = g}} m_{h_1,\ldots,h_n}(\lambda) a_{h_1}a_{h_2}\cdots a_{h_n}.
$$
With $n = 2$ and $g = 4$ we find

$$(\lambda_1 \alpha_1 + \lambda_2 \alpha_2)^{[4]}$$

$$= \lambda_1^4 \alpha_1^{[4]} + \lambda_2^4 \alpha_2^{[4]} + (\lambda_1^3 \lambda_2 \alpha_1^{[3]} \alpha_2 + \lambda_2^3 \lambda_1 \alpha_2^{[3]} \alpha_1) + \lambda_1^2 \lambda_2^2 \alpha_1^{[2]} \alpha_2^{[2]}.$$  

Applying $E$ this gives

$$(\lambda_1^4 + \lambda_2^4) a_0 a_4 + (\lambda_1^3 \lambda_2 + \lambda_2^3 \lambda_1) a_1 a_3 + \lambda_1^2 \lambda_2^2 a_2^2,$$

$$= m_{4,0}(\lambda) a_0 a_4 + m_{3,1}(\lambda) a_1 a_3 + m_{2,2}(\lambda) a_2^2.$$  

With $n = g = 3$ we find

$$E((\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 \alpha_3)^{[3]}) =$$

$$m_{3,0,0}(\lambda) a_0^2 a_3 + m_{2,1,0}(\lambda) a_0 a_1 a_2 + m_{1,1,1}(\lambda) a_1^3.$$
Duality

\[ \pi_{n,g}(\lambda; a) = E \left( \left( \sum_{r=1}^{n} \lambda_r \alpha_r \right)[g] \right) = \sum_{h_1 \geq \cdots \geq h_n \geq 0 \atop h_1 + \cdots + h_n = g} m_{h_1, \ldots, h_n}(\lambda) a_{h_1} a_{h_2} \cdots a_{h_n}. \]

\[ \pi_{n,g}(\lambda; a) \in \Sigma_{n,g} \otimes \mathbb{C}[a]_{n,g} \] is a dualizing tensor

that is gives a duality between symmetric functions in \( n \) variables of degree \( g \) and polynomials in the variables \( a_i \) of degree \( n \) and weight \( g \).
An elementary but not well known fact

Given two finite dimensional vector spaces $U, W$ and denoting by $U^\vee, W^\vee$ their duals one has the canonical isomorphisms

$$U \otimes W \simeq \text{hom}(U^\vee, W) \simeq \text{hom}(W^\vee, U).$$

A dualizing tensor $\pi \in U \otimes W$ is an element which corresponds, under these isomorphisms, to an isomorphism $U^\vee \simeq W, W^\vee \simeq U$.

Thus, a dualizing tensor $\pi$ equals, for any basis $u_1, \ldots, u_k$ of $U$, to

$$\pi = \sum_{i=1}^{k} u_i \otimes w_i,$$

where $w_1, \ldots, w_k$ is a basis of $W$. 

A basic Formula

From the main Formula of umbral calculus one has

\[ \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} \pi_{n,g} = D \pi_{n,g} = (\sum_{i=1}^{n} \lambda_i) \pi_{g-1,n} = e_1 \pi_{g-1,n} \]

The meaning of this formula is that, using the duality between symmetric functions in \( n \) variables and polynomials in the \( a_i \) of degree \( n \):

the transpose of the operator \( D \) is the multiplication by \( e_1 = \sum_{i=1}^{n} \lambda_i \).
Change the basis of the space \( \Sigma_{n,g} \) of symmetric functions from the total monomial sums \( m_{h_1,\ldots,h_n} \) to the monomials

\[
e_1^{k_1} \cdots e_n^{k_n}, \quad \sum_j jk_j = g,
\]

where \( e_i \) is the \( i \)\textsuperscript{th} elementary symmetric function.

\[
m_{h_1,\ldots,h_n} = \sum_{k_1,\ldots,k_n} \alpha_{h_1,\ldots,h_n;k_1,\ldots,k_n} e_1^{k_1} \cdots e_n^{k_n}. \tag{1}
\]

The \( \alpha_{h_1,\ldots,h_n,k_1,\ldots,k_n} \) are computable integers obtained inverting the obvious expansion

\[
e_1^{k_1} \cdots e_n^{k_n} = \sum_{h_1,\ldots,h_n} \beta_{h_1,\ldots,h_n;k_1,\ldots,k_n} m_{h_1,\ldots,h_n}. \tag{2}
\]
Example

\[ e_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n: \]

\[ e_1^k = \sum_{h_1, \ldots, h_n} \binom{k}{h_1, \ldots, h_n} m_{h_1, \ldots, h_n}. \]
The potenziante $\pi_{n,g}$ in this new basis appears as

$$
\pi_{n,g} = \sum_{0 \leq k_1, \ldots, k_n \atop k_1 + 2k_2 + \cdots + nk_n = g} e_1^{k_1} \cdots e_n^{k_n} \tilde{U}_{k_1, \ldots, k_n},
$$

where the polynomials $\tilde{U}_{k_1, \ldots, k_n}$ are given by

$$
\tilde{U}_{k_1, \ldots, k_n} = \sum_{h_1, \ldots, h_n} \alpha_{h_1, \ldots, h_n; k_1, \ldots, k_n} \prod_{j=1}^{n} a_{h_j},
$$

Example

$$
\pi_{2,4} = e_1^4 a_0 a_4 + e_2^2 (2a_0 a_4 - 2a_1 a_3 + a_2^2) + e_1^2 e_2 (a_1 a_3 - 4a_0 a_4).
$$
A basis of the $U$-invariants

We get from $D^{\pi_n, g} = (\sum_{i=1}^{n} \lambda_i) \pi_{n, g-1} = e_1 \pi_{n, g-1}$:

$$D^{\pi_n, g} = \sum_{k_1, \ldots, k_n \geq 0 \atop \sum i \cdot k_i = g} e_1^{k_1} \ldots e_n^{k_n} D^{\tilde{U}_{k_1, \ldots, k_n}} = \sum_{j_1, \ldots, j_n \geq 0 \atop \sum i \cdot j_i = g-1} e_1^{j_1+1} \ldots e_n^{j_n} \tilde{U}_{j_1, \ldots, j_n}$$

Example

$$\pi_{2,4} = e_1^4 a_0 a_4 + e_2^2 (2a_0 a_4 - 2a_1 a_3 + a_2^2) + e_1^2 e_2 (a_1 a_3 - 4a_0 a_4).$$

$$D^{\pi_{2,4}} = e_1^4 a_0 a_3 + e_1^2 e_2 (a_1 a_2 - 3a_0 a_3) = e_1 \pi_{2,3}.$$
A basis of the $U$-invariants

which implies

$$D\tilde{U}_{k_1,\ldots,k_n} = \begin{cases} 0 & \text{if } k_1 = 0 \\ \tilde{U}_{k_1-1,\ldots,k_n} & \text{if } k_1 > 0. \end{cases}$$

Example

$$D\tilde{U}_{0,2} = D(2a_0a_4 - 2a_1a_3 + a_2^2) = 0.$$

Theorem

The elements $U_{k_2,\ldots,k_n} := \tilde{U}_{0,k_2,\ldots,k_n}$, $\sum_{i=2}^n ik_i = g$ form a basis of the space $S_{n,g}$ of the $U$-invariants of degree $n$ and weight $g$. 
A duality for the \( U \)-invariants, set \( \sum_{i=1}^{n} \lambda_i = 0 \)

**Definition**

Denote by \( \Sigma_{n,g} \subset \mathbb{C}[\bar{\lambda}_1, \ldots, \bar{\lambda}_n] = \mathbb{C}[\lambda_1, \ldots, \lambda_n]/(\sum_{i=1}^{n} \lambda_i) \) the subspace of *symmetric polynomials* in \( \bar{\lambda}_1, \ldots, \bar{\lambda}_n \) which are homogeneous of degree \( g \) (observe that \( \bar{e}_1 = \sum_{i=1}^{n} \bar{\lambda}_i = 0 \)).

**Theorem**

The elements \( U_{k_2, \ldots, k_n} := \tilde{U}_{0,k_2, \ldots, k_n} \) form a basis of the space \( S_{n,g} \) of the \( U \)-invariants of degree \( n \) and weight \( g \) dual, via \( \tilde{\pi}_{n,g} \), to the basis \( e^ {k_2}_2 \ldots e^{k_n}_n \) of \( \Sigma_{n,g} \).
A duality for the $U$-invariants

$$\sum_{g=0}^{\infty} \dim(\Sigma_{n,g}) x^g = \sum_{g=0}^{\infty} \dim(S_{n,g}) x^g = \frac{1}{(1 - x^2)(1 - x^3) \cdots (1 - x^n)}$$
Our Main Theorem, \textit{development of Stroh}

we use the partial order

$$\left( t_2, \ldots, t_n \right) \succeq \left( s_2, \ldots, s_n \right) \iff t_i \geq s_i \text{ for all } i.$$ 

\textbf{Theorem}

\textit{The elements} \( U_k = U_{k_2, \ldots, k_n} = \tilde{U}_{0, k_2, \ldots, k_n} \) \textit{with} \( \sum_{i=2}^{n} i \cdot k_i = g \) \textit{and}

$$k \succeq n = (0, 2^{n-4}, 2^{n-5}, \ldots, 4, 2, 1, 1), \quad (\text{resp. } n = (0, 1))$$

\textit{form a basis of a space of perpetuants of degree} \( n > 3 \) \textit{(resp.} \( n = 3 \)) \textit{and weight} \( g \).

\textit{For} \( n = 2 \) \textit{one perpetuant,} \( U_{2k} \), \textit{in each even weight} \( 2k > 0 \).
\textit{For} \( n = 1 \) \textit{just } \( a_0 \).
Notice

It is a remarkable fact that we are unable to exhibit minimal sets of generators for $U$ invariants of forms of a given degree $q$ but in fact we are able to do this for the limit, infinite, case.
The ideas of Stroh

A sketch of the proof
A sketch of the proof

the main idea is to describe, in the duality between symmetric functions and $U$ invariants, the:

space of symmetric functions orthogonal to the space of decomposable elements.

The decomposable elements of $S_{n,g}$ are

$$
\sum_{1\leq h \leq n/2} S_{n,g,h}, \quad S_{n,g,h} := \sum_{j=0}^{g} S_{h,j} \cdot S_{n-h,g-j}.
$$
A sketch of the proof

For a given $h \in \mathbb{N}$, $1 \leq h \leq n/2$ we have:

$$(\lambda_1 \alpha_1 + \cdots + \lambda_n \alpha_n)^[g] = \sum_{j=0}^{g} (\lambda_1 \alpha_1 + \cdots + \lambda_h \alpha_h)^[j] (\lambda_{h+1} \alpha_{h+1} + \cdots + \lambda_n \alpha_n)^[g-j]$$

which implies the following decomposition of the potenziante:

$$\pi_{n,g}(\lambda_1, \ldots, \lambda_n; a) = \sum_{j=0}^{g} \pi_{h,j}(\lambda_1, \ldots, \lambda_h; a) \cdot \pi_{n-h,g-j}(\lambda_{h+1}, \ldots, \lambda_n; a).$$
A sketch of the proof

Consider the ideal $J_h \subset \mathbb{C}[\lambda_1, \ldots, \lambda_n]$ generated by the two elements $\lambda_1 + \cdots + \lambda_h$ and $\lambda_{h+1} + \cdots + \lambda_n$.

Main remark

Modulo this ideal the potenziante $\pi_{n,g}$ becomes a dualizing element between the image of $\Sigma_{n,g}$ and $S_{n,g,h}$. 
This implies that

the orthogonal to $S_{n,g,h}$ is the space of elements of $\Sigma_{n,g}$ which are multiples of the symmetric function

$$p_h := \prod_{1 \leq j_1 < j_2 < \ldots < j_h \leq n} (\bar{\lambda}_{j_1} + \bar{\lambda}_{j_2} + \cdots + \bar{\lambda}_{j_h})$$
A sketch of the proof

1. The orthogonal $O_{n,g}$ to the decomposable elements $\sum_{1 \leq h \leq n/2} S_{n,g,h}$ of $S_{n,g}$ is thus the intersection of these orthogonals,

2. so $O_{n,g}$ is the space of elements of $\overline{\sum}_{n,g}$ which are multiples of all the symmetric functions $p_h$, $1 \leq h \leq n/2$

3. but these are irreducible elements in the algebra of symmetric functions so a common multiple is a multiple of their product!
A sketch of the proof

**Summarizing**

The orthogonal to the decomposable elements $\sum_{1 \leq h \leq n/2} S_{n,g,h}$ of $S_{n,g}$ is the space $O_{n,g}$ of elements of $\bar{\Sigma}_{n,g}$ which are multiples of the product

$$q_n := \prod_{h} p_h, \quad \deg q_n = 2^{n-1} - 1.$$

This space equals

$$O_{n,g} = q_n \cdot \bar{\Sigma}_{n,g-2^{n-1}+1}$$

$$\dim O_{n,g} = \dim \bar{\Sigma}_{n,g-2^{n-1}+1} = \dim (P_{n,g})$$

from which Stroh’s Theorem holds.

This is the main step, the next is to analyze a basis of the complement of this space of multiples.
The final Theorem

This is done by using leading exponents, of polynomials in $\bar{\lambda}_1, \ldots, \bar{\lambda}_{n-1}$ and proving, by duality, the stated theorem:

For $n \geq 4$ we have the leading exponents $le$ of $q_n$ and of $e^n$ are the same

$$le = (2^{n-2}, 2^{n-3}, \ldots, 2, 1) \quad \text{where } n := (0, 2^{n-4}, 2^{n-5}, \ldots, 2, 1, 1).$$

$$q_n = \bar{\lambda}_1^{2^{n-2}} \bar{\lambda}_2^{2^{n-3}} \ldots \bar{\lambda}_{n-1} + \text{lower terms}$$

$$e^n = e_3^{2^{n-4}} e_4^{2^{n-5}} \ldots e_{n-1} e_n = \bar{\lambda}_1^{2^{n-2}} \bar{\lambda}_2^{2^{n-3}} \ldots \bar{\lambda}_{n-1} + \text{lower terms}$$
The final Theorem

Theorem

The elements $U_k = U_{k_2,...,k_n} = \tilde{U}_{0,k_2,...,k_n}$ with

$$k \succeq n = (0, 2^{n-4}, 2^{n-5}, \ldots, 4, 2, 1, 1)$$

(resp. $n = (0, 1)$) form a basis of a space of perpetuants of degree $n > 3$ (resp. $n = 3$) and weight $g$.

The proof is by showing that the products of elementary functions in the dual basis to the $U_k = U_{k_2,...,k_n}$ for $k \not\preceq n = (0, 2^{n-4}, \ldots, 2, 1, 1)$ form a basis of a complement of $O_{n,g}$. This is done by looking at the leading exponents.
Reference

arXiv:1810.01131 math.AG math.AC
Perpetuants: A Lost Treasure
Authors: Hanspeter Kraft, Claudio Procesi