Character triples and group graded equivalences

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Motivation

Categorical version of reduction theorems involving character triples.

Assumptions and notations

- $G$ is a finite group
- $(\mathcal{K}, \mathcal{O}, k)$ is a splitting $p$-modular system
- $N \trianglelefteq G$, $G' \leq G$, and $N' \trianglelefteq G'$
  
  Assume: $N' = G' \cap N$ and $G = G'N$, hence $\bar{G} := G/N \simeq G'/N'$
- $b \in Z(\mathcal{O}N)$ and $b' \in Z(\mathcal{O}N')$ are $\bar{G}$-invariant blocks
- $A := b\mathcal{O}G$ and $A' := b'\mathcal{O}G'$, strongly $\bar{G}$-graded algebras with 1-components $B = b\mathcal{O}N$ and $B' = b'\mathcal{O}N'$
- $C_G(N) \subseteq G'$
- $C := \mathcal{O}C_G(N)$
**Definition**

An algebra $C$ is a $\tilde{G}$-graded $\tilde{G}$-acted algebra if

1. $C$ is $\tilde{G}$-graded, i.e. $C = \bigoplus_{g \in \tilde{G}} C_{\bar{g}}$;
2. $\tilde{G}$ acts on $C$ (always on the left in this presentation);
3. $\forall \bar{h} \in \tilde{G}$, $\forall c \in C_{\bar{h}}$, we have that $\bar{g}c \in C_{\bar{g}\bar{h}}$ for all $\bar{g} \in \tilde{G}$.

**Remark**

$C := \mathcal{O}C_G(N)$ is a $\tilde{G}$-graded $\tilde{G}$-acted algebra. Moreover, there exists two $\tilde{G}$-graded $\tilde{G}$-acted algebra homomorphisms $\zeta : C \to C_A(B)$ and $\zeta' : C \to C_{A'}(B')$, i.e. for any $\bar{h} \in \tilde{G}$ and $c \in C_A(B)_{\bar{h}}$, we have $\zeta(c) \in C_A(B)_{\bar{h}}$ and $\zeta'(c) \in C_{A'}(B')_{\bar{h}}$, and for every $\bar{g} \in \tilde{G}$, $\zeta(\bar{g}c) = \bar{g}\zeta(c)$ and $\zeta'(\bar{g}c) = \bar{g}\zeta'(c)$. 
**Definition**

We say that $\tilde{M}$ is a $\tilde{G}$-graded $(A, A')$-bimodule over $C$ if:

1. $\tilde{M}$ is an $(A, A')$-bimodule;
2. $\tilde{M}$ has a decomposition $\tilde{M} = \bigoplus_{g \in \tilde{G}} \tilde{M}_g$ such that $A\tilde{g} \tilde{M}_{\tilde{x}} A' \subseteq \tilde{M}_{\tilde{g}\tilde{x}\tilde{h}}$, for all $\tilde{g}, \tilde{x}, \tilde{h} \in \tilde{G}$;
3. $\tilde{m}_g \cdot c = \tilde{g}c \cdot \tilde{m}_g$, for all $c \in C, \tilde{m}_g \in \tilde{M}_g, \tilde{g} \in \tilde{G}$, where $c \cdot \tilde{m} = \zeta(c) \cdot \tilde{m}$ and $\tilde{m} \cdot c = \tilde{m} \cdot \zeta'(c)$, for all $c \in C, \tilde{m} \in \tilde{M}$.

**Remark**

Note that homomorphisms between $\tilde{G}$-graded $(A, A')$-bimodules over $C$ are just homomorphism between $\tilde{G}$-graded $(A, A')$-bimodules.
We say that a \( \tilde{G} \)-graded \((A, A')\)-bimodule over \( C \), \( \tilde{M} \), induces a \( \tilde{G} \)-graded Morita equivalence over \( C \) between \( A \) and \( A' \), if \( \tilde{M} \otimes_{A'} \tilde{M}^* \cong A \) as \( \tilde{G} \)-graded \((A, A)\)-bimodules over \( C \) and that \( \tilde{M}^* \otimes_A \tilde{M} \cong A' \) as \( \tilde{G} \)-graded \((A', A')\)-bimodules over \( C \), where the \( A \)-dual \( \tilde{M}^* = \text{Hom}_A(\tilde{M}, A) \) of \( \tilde{M} \) is a \( \tilde{G} \)-graded \((A', A)\)-bimodule.
We regard $A^{\text{op}}$ as a $\tilde{G}$-graded algebra with components $(A^{\text{op}})_{\bar{g}} = A'_{\bar{g}^{-1}}$, $\forall \bar{g} \in \tilde{G}$. We denote by $\ast$ the multiplicative operations in $A^{\text{op}}$. We also define the $(\bar{g}, \bar{h})$ component of $(A \otimes_C A^{\text{op}})_{(\bar{g}, \bar{h})} := A_{\bar{g}} \otimes_C A'_{\bar{h}}$. Let

$$\delta(\tilde{G}) := \{(\bar{g}, \bar{g}) \mid \bar{g} \in \tilde{G}\}$$

be the diagonal subgroup of $\tilde{G} \times \tilde{G}$, and let $\Delta^C$ be the diagonal part of $A \otimes_C A^{\text{op}}$:

$$\Delta^C := \Delta(A \otimes_C A^{\text{op}}) := (A \otimes_C A^{\text{op}})_{\delta(\tilde{G})} = \bigoplus_{\bar{g} \in \tilde{G}} A_{\bar{g}} \otimes_C A'_{\bar{g}^{-1}},$$

which clearly has the 1-component defined as follows:

$$\Delta^C_1 = B \otimes_C B^{\text{op}}.$$
Lemma

$\Delta^C$ is an $\mathcal{O}$-algebra and there exists an $\mathcal{O}$-algebra homomorphism from $C$ to $\Delta^C$:

$$\varphi : C \rightarrow \mathbb{Z}(\Delta^C_1), \varphi(c) := \zeta(c) \otimes_c 1 = 1 \otimes_c \zeta'(c).$$

Lemma

$A \otimes_C A'^{\text{op}}$ is a right $\Delta^C$-module and a $\check{G}$-graded $(A, A')$-bimodule over $C$.

Lemma

Let $M$ be a $\Delta^C$-module, then

$$A \otimes_B M, M \otimes_{B'} A', (A \otimes_C A'^{\text{op}}) \otimes_{\Delta^C} M$$

are isomorphic as $\check{G}$-graded $(A, A')$-bimodules over $C$. We shall denote them by $\tilde{M}$.
Lemma 1 Let $M$ be a $\Delta(A \otimes_C A'^{op})$-module and $M'$ be a $\Delta(A' \otimes_C A''^{op})$-module. Then $M \otimes_{B'} M'$ is a $\Delta(A \otimes_C A''^{op})$-module with the multiplication operation defined as follows:

$$(a_{\bar{g}} \otimes c a''^{op}_{\bar{g}^{-1}})(m \otimes_{B'} m') := (a_{\bar{g}} \otimes c (u_{\bar{g}}'^{-1})^{op}) m \otimes_{B'} (u_{\bar{g}}' \otimes c a''^{op}_{\bar{g}^{-1}}) m'$$

for all $\bar{g} \in \tilde{G}$, $a_{\bar{g}} \in A_{\bar{g}}$, $a''^{op}_{\bar{g}^{-1}} \in A''^{op}_{\bar{g}^{-1}}$, $m \in M$, $m' \in M'$. Moreover, we have the isomorphism

$$\tilde{M} \otimes_{B'} \tilde{M}' \cong \tilde{M} \otimes_{A'} \tilde{M}'$$

of $\tilde{G}$-graded $(A, A'')$-bimodules over $C$. 
Lemma

Let $M$ be a $\Delta(A' \otimes_C A'^{op})$-module and $M'$ be a $\Delta(A' \otimes_C A''^{op})$-module. Then $\text{Hom}_{B'}(M, M')$ is a $\Delta(A \otimes_C A''^{op})$-module with the following operation:

$$(a_{\tilde{g}} f a'_{\tilde{g}-1})(m) := (u_{\tilde{g}} \otimes_C (a''_{\tilde{g}-1})^{op}) f ((u_{\tilde{g}}^{-1} \otimes_C a_{\tilde{g}}^{op})m)$$

for all $\tilde{g} \in \tilde{G}$ and for all $a_{\tilde{g}} \in A_{\tilde{g}}$, $a''_{\tilde{g}-1} \in A''_{\tilde{g}-1}$, $m \in M$, $f \in \text{Hom}_{B'}(M, M')$. Moreover, we have the isomorphism

$$\text{Hom}_{B'}(\tilde{M}, \tilde{M}') \cong \text{Hom}_{A'}(\tilde{M}, \tilde{M}')$$

of $\tilde{G}$-graded $(A, A'')$-bimodules over $C$. 
Let $B\mathcal{M}_{B'}$ and $B'\mathcal{M}^*_B := \text{Hom}_B(M, B)$ (the $B$-dual of $M$) be two bimodules that induce a Morita equivalence between $B$ and $B'$:

$$B \xleftarrow{\mathcal{M}^* \otimes B^-} \xrightarrow{M \otimes B'} B'$$

If $M$ extends to a $\Delta^C$-module, then we have the following:

1. $M^*$ becomes a $\Delta(A' \otimes_C A^{op})$-module;
2. $\mathcal{M} := (A \otimes_C A'^{op}) \otimes_{\Delta C} M$ and $\tilde{\mathcal{M}}^* := (A' \otimes_C A^{op}) \otimes_{\Delta} (A' \otimes_C A^{op})$

$M^*$ are $\tilde{G}$-graded $(A, A')$-bimodules over $C$ and they induce a $\tilde{G}$-graded Morita equivalence over $C$ between $A$ and $A'$:

$$A \xleftarrow{\sim} \xrightarrow{} A'.$
In this section, we attempt to give a version with Morita equivalences for the relationship \( \leq_c \) given in [2, Definition 2.7.].

**Proposition**

Let \( A \) and \( A' \) be two strongly \( \tilde{G} \)-graded algebras over \( C \). Assume that \( \tilde{M} \) is a \( \tilde{G} \)-graded \( (A, A') \)-bimodule over \( C \), which induces a Morita equivalence between \( A \) and \( A' \). Let \( U \) be a (left) \( B \)-module and let \( U' \) be a (left) \( B' \)-module corresponding to \( U \) under the given equivalence. Then there is a commutative diagram:

\[
\begin{array}{ccc}
E(U) & \sim & E(U') \\
\uparrow & & \uparrow \\
C_A(B) & \sim & C_A'(B') \\
\uparrow & & \uparrow \\
C & id_C & C.
\end{array}
\]
Definition

Let $V$ be a $G$-invariant simple $\mathcal{K}B$-module, $V'$ a $G'$-invariant simple $\mathcal{K}B'$-module. We say that $(A, B, V) \succeq_c (A', B', V')$ if

1. $G = G'N$, $N' = N \cap G'$
2. $C_G(N) \subseteq G'$
3. we have the following commutative diagram of $\tilde{G}$-graded $\mathcal{K}$-algebras:

\[
\begin{array}{ccc}
E(V) & \sim & E(V') \\
\uparrow & & \uparrow \\
\mathcal{K}C & \xrightarrow{id_C} & \mathcal{K}C.
\end{array}
\]

where $\mathcal{K}C = \mathcal{K}C_G(N)$ is regarded as a $\tilde{G}$-graded $\tilde{G}$-acted $\mathcal{K}$-algebra, with 1-component $\mathcal{K}Z(N)$. 
Proposition

Assume that \( \tilde{M} \) induces a \( \tilde{G} \)-graded Morita equivalence over \( C := \mathcal{O}C_G(N) \) between \( A \) and \( A' \). Let \( V \) be a simple \( KB \)-module and \( V' \) be a simple \( KB' \)-module corresponding to \( V' \) via the given correspondence. Then we have that \( (A, B, V) \geq_c (A', B', V') \).

Proposition

Let \( \theta \) be the character associated to \( V \) and \( \theta' \) the character associated to \( V' \). If \( (A, B, V) \geq_c (A', B', V') \), then \( (G, N, \theta) \geq_c (G', N', \theta') \).
Butterfly theorem

Let $\hat{G}$ be another group with normal subgroup $N$. Assume that:

1. $C_G(N) \subseteq G'$,
2. $\tilde{M}$ induces a $\tilde{G}$-graded Morita equiv. over $C$ between $A$ and $A'$;
3. the conjugation maps $\varepsilon : G \to \text{Aut}(N)$ and $\hat{\varepsilon} : \hat{G} \to \text{Aut}(N)$ satisfy $\varepsilon(G) = \hat{\varepsilon}(\hat{G})$.

Denote $\hat{G}' = \hat{\varepsilon}^{-1}(\varepsilon(G'))$. Then there is a $\hat{G}/N$-graded Morita equivalence over $\hat{C} := \hat{O}C_{\hat{G}}(N)$ between $\hat{A} := b\hat{O}\hat{G}$ and $\hat{A}' := b'\hat{O}\hat{G}'$.

\[
\begin{align*}
\hat{A} &:= b\hat{O}\hat{G} & A &:= b\hat{O}\hat{G} \sim \tilde{M} & A' &:= b'\hat{O}\hat{G}' & \hat{A}' &:= b'\hat{O}\hat{G}' \\
b\hat{O}NC_{\hat{G}}(N) &\sim b\hat{O}NC_{\hat{G}}(N) \sim b'O'\hat{N}'C_{\hat{G}}(N) & b\hat{O}NC_{\hat{G}}(N) &\sim b'O'\hat{N}'C_{\hat{G}}(N) & b'O'\hat{N}'C_{\hat{G}}(N) & \\
B &:= \hat{O}\hat{N}b \sim M & B' &:= \hat{O}\hat{N}'b'.
\end{align*}
\]