Group rings and the RS property

Sugandha Maheshwary
(Joint work with G.K. Bakshi and I.B.S. Passi)

Indian Institute of Science Education & Research (IISER), Mohali
INDIA
sugandha@iisermohali.ac.in

June 10, 2019
Introduction

- $G$: a group (not necessarily finite).
- $\mathcal{Z}(G)$: centre of $G$
- $\mathcal{U}(\mathbb{Z}G)$: the group of units of the integral group ring $\mathbb{Z}G$
- $V(\mathbb{Z}G)$: the group of normalized units in $\mathbb{Z}G$

Clearly, $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ contains $\pm \mathcal{Z}(G)$, the trivial units in $\mathbb{Z}G$.

Question

When is $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) = \pm \mathcal{Z}(G)$? i.e., When are all the central units of $\mathbb{Z}G$ trivial?
\[ G: \text{ a group (not necessarily finite).} \]
\[ \mathcal{Z}(G): \text{ centre of } G \]
\[ \mathcal{U}(\mathbb{Z}G): \text{ the group of units of the integral group ring } \mathbb{Z}G \]
\[ \mathcal{V}(\mathbb{Z}G): \text{ the group of normalized units in } \mathbb{Z}G \]

### Trivial units

Clearly, \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \) contains \( \pm \mathcal{Z}(G) \), the trivial units in \( \mathbb{Z}G \).
$G$: a group (not necessarily finite).
$\mathcal{Z}(G)$: centre of $G$
$\mathcal{U}(\mathbb{Z}G)$: the group of units of the integral group ring $\mathbb{Z}G$
$\mathcal{V}(\mathbb{Z}G)$: the group of normalized units in $\mathbb{Z}G$

**Trivial units**
Clearly, $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ contains $\pm \mathcal{Z}(G)$, the trivial units in $\mathbb{Z}G$.

**Question**
When is $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) = \pm \mathcal{Z}(G)$? i.e.,
When are all the central units of $\mathbb{Z}G$ trivial?
Timeline

[9x252]Timeline

[Hig40]: All central units of $\mathbb{Z}_G$, $G$ abelian, are trivial if and only if, the exponent of group divides 4 or 6.

[CL65]: Trivial units in $\mathbb{R}G$, central units of finite order are trivial.

[GP86]: Pointed, no necessary and sufficient condition for all central units in a group ring to be trivial, yet.

[Bov87]: All central units of $\mathbb{Z}_G$ are trivial, if and only if, the fields involved in decomposition of $\mathbb{Q}G$ are only $\mathbb{Q}$ and imaginary quadratic.

[RS90]: For a finite group $G$, all central units of $\mathbb{Z}_G$ are trivial, if and only if, for every $x$ in $G$, and for every natural number $j$, relatively prime to $|G|$, the order of $G$, $x^j$ is conjugate in $G$ to $x$ or $x^{-1}$.

[Seh93]: A classification of arbitrary groups $G$ with only trivial central units in $\mathbb{Z}_G$ was asked for (Problem 26).

[Par99] Pointed above question again, in survey entitled "Central Units in Integral Group Rings".

S.Maheshwary (IISERM)
[Hig40]: All central units of $\mathbb{Z}G$, $G$ abelian, are trivial if and only if, the exponent of group divides 4 or 6.
[Hig40]: All central units of $\mathbb{Z}G$, $G$ abelian, are trivial if and only if, the exponent of group divides 4 or 6.

[CL65]: Trivial units in $RG$, central units of finite order are trivial.
[Hig40]: All central units of $\mathbb{Z}G$, $G$ abelian, are trivial if and only if, the exponent of group divides 4 or 6.

[CL65]: Trivial units in $RG$, central units of finite order are trivial.

[GP86]: Pointed, no necessary and sufficient condition for all central units in a group ring to be trivial, yet.
[Hig40]: All central units of $\mathbb{Z}G$, $G$ abelian, are trivial if and only if, the exponent of group divides 4 or 6.

[CL65]: Trivial units in $RG$, central units of finite order are trivial.

[GP86]: Pointed, no necessary and sufficient condition for all central units in a group ring to be trivial, yet.

[Bov87]: All central units of $\mathbb{Z}G$ are trivial, if and only if, the fields involved in decomposition of $\mathbb{Q}G$ are only $\mathbb{Q}$ and imaginary quadratic.

[RS90]: For a finite group $G$, all central units of $\mathbb{Z}G$ are trivial, if and only if, for every $x$ in $G$, and for every natural number $j$, relatively prime to $|G|$, the order of $G$, $x^j$ is conjugate in $G$ to $x$ or $x-1$.

[Seh93]: A classification of arbitrary groups $G$ with only trivial central units in $\mathbb{Z}G$ was asked for (Problem 26).

[Par99] Pointed above question again, in survey entitled "Central Units in Integral Group Rings".
[Hig40]: All central units of \( \mathbb{Z}G \), \( G \) abelian, are trivial if and only if, the exponent of group divides 4 or 6.

[CL65]: Trivial units in \( RG \), central units of finite order are trivial.

[GP86]: Pointed, no necessary and sufficient condition for all central units in a group ring to be trivial, yet.

[Bov87]: All central units of \( \mathbb{Z}G \) are trivial, if and only if, the fields involved in decomposition of \( \mathbb{Q}G \) are only \( \mathbb{Q} \) and imaginary quadratic.

[RS90]: For a finite group \( G \), all central units of \( \mathbb{Z}G \) are trivial, if and only if, for every \( x \) in \( G \), and for every natural number \( j \), relatively prime to \(|G|\), the order of \( G \), \( x^j \) is conjugate in \( G \) to \( x \) or \( x^{-1} \).
Timeline

- [Hig40]: All central units of $\mathbb{Z}G$, $G$ abelian, are trivial if and only if, the exponent of group divides 4 or 6.
- [CL65]: Trivial units in $RG$, central units of finite order are trivial.
- [GP86]: Pointed, no necessary and sufficient condition for all central units in a group ring to be trivial, yet.
- [Bov87]: All central units of $\mathbb{Z}G$ are trivial, if and only if, the fields involved in decomposition of $\mathbb{Q}G$ are only $\mathbb{Q}$ and imaginary quadratic.
- [RS90]: For a finite group $G$, all central units of $\mathbb{Z}G$ are trivial, if and only if, for every $x$ in $G$, and for every natural number $j$, relatively prime to $|G|$, the order of $G$, $x^j$ is conjugate in $G$ to $x$ or $x^{-1}$.
- [Seh93]: A classification of arbitrary groups $G$ with only trivial central units in $\mathbb{Z}G$ was asked for (Problem 26).
[Hig40]: All central units of $\mathbb{Z}G$, $G$ abelian, are trivial if and only if, the exponent of group divides 4 or 6.

[CL65]: Trivial units in $RG$, central units of finite order are trivial.

[GP86]: Pointed, no necessary and sufficient condition for all central units in a group ring to be trivial, yet.

[Bov87]: All central units of $\mathbb{Z}G$ are trivial, if and only if, the fields involved in decomposition of $\mathbb{Q}G$ are only $\mathbb{Q}$ and imaginary quadratic.

[RS90]: For a finite group $G$, all central units of $\mathbb{Z}G$ are trivial, if and only if, for every $x$ in $G$, and for every natural number $j$, relatively prime to $|G|$, the order of $G$, $x^j$ is conjugate in $G$ to $x$ or $x^{-1}$.

[Seh93]: A classification of arbitrary groups $G$ with only trivial central units in $\mathbb{Z}G$ was asked for (Problem 26).

[Par99] Pointed above question again, in survey entitled “Central Units in Integral Group Rings".
Timeline

[Fer04]: All central units of $\mathbb{Z}_n$, are trivial if, and only if $n \in \{1, 2, 3, 7, 8, 9, 12\}$.

[JJdMR02]: For an arbitrary group $G$, all central units of $\mathbb{Z}_G$ are trivial, if and only if $G = \text{N}_V(G)$, the normalizer of $G$ in $V$.

[DMS05]: For an arbitrary group $G$, the central units of $\mathbb{Z}_G$ are trivial, if and only if, every finite normal subgroup $A$ of $G$ satisfies (RS) in $G$:

$\forall a \in A$ and $j \in \mathbb{Z}$, with $(j, |a|) = 1$, we have that $a^j \sim G a^\pm 1$.

(RS)

[CD10]: Studied the class of inverse semi rational groups, which turns out to be the class of finite groups $G$ with all central units of $\mathbb{Z}_G$ trivial.

[BMP17]: The groups $G$ for which central units of $\mathbb{Z}_G$ trivial, were called as cut groups, or the groups with the cut property. Properties of finite cut groups, especially $p$-groups, explored. Classified finite metacyclic cut groups.
[Fer04]: All central units of $\mathbb{Z}A_n$, are trivial if, and only if $n \in \{1, 2, 3, 7, 8, 9, 12\}$. 

[JJdMR02]: For an arbitrary group $G$, all central units of $\mathbb{Z}G$ are trivial, if and only if $G = N_V(G)$, the normalizer of $G$ in $V$.

[DMS05]: For an arbitrary group $G$, the central units of $\mathbb{Z}G$ are trivial, if and only if, every finite normal subgroup $A$ of $G$ satisfies (RS) in $G$: $\forall a \in A$ and $j \in \mathbb{Z}$, with $(j, |a|) = 1$, we have that $a^j \sim_G a \pm 1$. (RS)

[CD10]: Studied the class of inverse semi rational groups, which turns out to be the class of finite groups $G$ with all central units of $\mathbb{Z}G$ trivial.

[BMP17]: The groups $G$ for which central units of $\mathbb{Z}G$ trivial, were called as cut groups, or the groups with the cut property. Properties of finite cut groups, especially $p$-groups, explored. Classified finite metacyclic cut groups.
[Fer04]: All central units of \( \mathbb{Z}A_n \), are trivial if, and only if 
\( n \in \{1,2,3,7,8,9,12\} \).

[JJdMR02]: For an arbitrary group \( G \), all central units of \( \mathbb{Z}G \) are 
trivial, if and only if \( G = \mathcal{N}_\mathcal{V}(G) \), the normalizer of \( G \) in \( \mathcal{V} \).

[DMS05]: For an arbitrary group \( G \), the central units of \( \mathbb{Z}G \) are 
trivial, if and only if, every finite normal subgroup \( A \) of \( G \) satisfies 
\( (RS) \) in \( G \):
\[
\forall a \in A \text{ and } j \in \mathbb{Z}, \text{ with } (j,|a|) = 1, \text{ we have that } a^j \sim_G a^\pm 1.\]

[CD10]: Studied the class of inverse semi rational groups, which turns 
out to be the class of finite groups \( G \) with all central units of \( \mathbb{Z}G \) trivial.

[BMP17]: The groups \( G \) for which central units of \( \mathbb{Z}G \) trivial, were 
called as cut groups, or the groups with the cut property. Properties 
of finite cut groups, especially \( p \)-groups, explored. Classified finite 
metacyclic cut groups.
Timeline

- [Fer04]: All central units of $\mathbb{ZA}_n$, are trivial if, and only if $n \in \{1, 2, 3, 7, 8, 9, 12\}$.
- [JJdMR02]: For an arbitrary group $G$, all central units of $\mathbb{Z}G$ are trivial, if and only if $G = \mathcal{N}_\mathcal{V}(G)$, the normalizer of $G$ in $\mathcal{V}$.
- [DMS05]: For an arbitrary group $G$, the central units of $\mathbb{Z}G$ are trivial, if and only if, every finite normal subgroup $A$ of $G$ satisfies (RS) in $G$:

\[ \forall a \in A \text{ and } j \in \mathbb{Z}, \text{ with } (j, |a|) = 1, \text{ we have that, } a^j \sim_G a^{\pm 1}. \quad (RS) \]
[Fer04]: All central units of $\mathbb{ZA}_n$, are trivial if, and only if $n \in \{1, 2, 3, 7, 8, 9, 12\}$.

[JJdMR02]: For an arbitrary group $G$, all central units of $\mathbb{Z}G$ are trivial, if and only if $G = N_V(G)$, the normalizer of $G$ in $V$.

[DMS05]: For an arbitrary group $G$, the central units of $\mathbb{Z}G$ are trivial, if and only if, every finite normal subgroup $A$ of $G$ satisfies (RS) in $G$: 
\[
\forall a \in A \text{ and } j \in \mathbb{Z}, \text{ with } (j, |a|) = 1, \text{ we have that } , a^j \sim_G a^{\pm 1}. \quad (RS)
\]

[CD10]: Studied the class of inverse semi rational groups, which turns out to be the class of finite groups $G$ with all central units of $\mathbb{Z}G$ trivial.
[Fer04]: All central units of $\mathbb{ZA}_n$, are trivial if, and only if $n \in \{1, 2, 3, 7, 8, 9, 12\}$.

[JJdMR02]: For an arbitrary group $G$, all central units of $\mathbb{Z}G$ are trivial, if and only if $G = \mathcal{N}_V(G)$, the normalizer of $G$ in $V$.

[DMS05]: For an arbitrary group $G$, the central units of $\mathbb{Z}G$ are trivial, if and only if, every finite normal subgroup $A$ of $G$ satisfies (RS) in $G$:

$$\forall a \in A \text{ and } j \in \mathbb{Z}, \text{ with } (j, |a|) = 1, \text{ we have that } a^j \sim_G a^{\pm 1}. \quad (RS)$$

[CD10]: Studied the class of inverse semi rational groups, which turns out to be the class of finite groups $G$ with all central units of $\mathbb{Z}G$ trivial.

[BMP17]: The groups $G$ for which central units of $\mathbb{Z}G$ trivial, were called as cut groups, or the groups with the cut property. Properties of finite cut groups, especially $p$-groups, explored. Classified finite metacyclic cut groups.
Timeline


[Bäc18]: Prime spectrum of finite solvable cut-groups, Frobenius cut groups.

[BCJM]: Global and local properties of finite cut groups are studied.

[BMP19]: Group Rings and the RS property, On arbitrary cut groups.
[Mah18]: Finite nilpotent and solvable cut-groups.
Timeline

- [Mah18]: Finite nilpotent and solvable cut-groups.
- [Bäc18]: Prime spectrum of finite solvable cut-groups, Frobenius cut groups.
Timeline

- [Mah18]: Finite nilpotent and solvable cut-groups.
- [Bäc18]: Prime spectrum of finite solvable cut-groups, Frobenius cut groups.
- [BCJM]: Global and local properties of finite cut groups are studied.
Timeline

- [Mah18]: Finite nilpotent and solvable cut-groups.
- [Bäc18]: Prime spectrum of finite solvable cut-groups, Frobenius cut groups.
- [BCJM]: Global and local properties of finite cut groups are studied.
- [BMP19]: Group Rings and the RS property, On arbitrary cut groups.
Definition (RS-property)
A torsion element $x \in G$, has the RS-property (or is an RS-element) in $G$, if $x^j \sim_G x \pm 1$ for all $j \in U(o(x))$, where $o(x)$ denotes the order of $x$, $U(n) := \{j : 1 \leq j \leq n, \gcd(j,n) = 1\}$, and $y \sim_G z$ denotes $y$ is conjugate to $z$ in $G$.

RS-subgroup
A subgroup $A$ of $G$ is an RS-subgroup of $G$, if every torsion element of $A$ is an RS-element in $G$.

Theorem ([DMS05], Theorems 8 & 9)
If $A$ is a normal subgroup of a finite group $G$, then $A$ is an RS-subgroup of $G$ if, and only if, $Z(U(Z[G])) \cap Z[A]$ consists of trivial units.
A torsion element $x \in G$, has the \textit{RS-property} (or is an \textit{RS-element}) in $G$, if

$$x^j \sim_G x^{\pm 1} \text{ for all } j \in U(o(x)),$$

where $o(x)$ denotes the order of $x$, $U(n) := \{ j : 1 \leq j \leq n, \gcd(j, n) = 1 \}$, and $y \sim_G z$ denotes $y$ is conjugate to $z$ in $G$. 

**Definition (RS-property)**

A torsion element $x \in G$, has the \textit{RS-property} (or is an \textit{RS-element}) in $G$, if

$$x^j \sim_G x^{\pm 1} \text{ for all } j \in U(o(x)),$$

where $o(x)$ denotes the order of $x$, $U(n) := \{ j : 1 \leq j \leq n, \gcd(j, n) = 1 \}$, and $y \sim_G z$ denotes $y$ is conjugate to $z$ in $G$. 

**Theorem ([DMS05], Theorems 8 & 9)**

If $A$ is a normal subgroup of a finite group $G$, then $A$ is an RS-subgroup of $G$ if, and only if, $Z(U(Z[G])) \cap Z[A]$ consists of trivial units.
Definition (RS-property)

A torsion element \( x \in G \), has the RS-property (or is an RS-element) in \( G \), if

\[ x^j \sim_G x^{\pm 1} \text{ for all } j \in U(o(x)), \]

where \( o(x) \) denotes the order of \( x \), \( U(n) := \{ j : 1 \leq j \leq n, \gcd(j, n) = 1 \} \), and \( y \sim_G z \) denotes \( y \) is conjugate to \( z \) in \( G \).

RS-subgroup

A subgroup \( A \) of \( G \) is an RS-subgroup of \( G \), if every torsion element of \( A \) is an RS-element in \( G \).
Definition (RS-property)

A torsion element $x \in G$, has the *RS-property* (or is an *RS-element*) in $G$, if

$$x^j \sim_G x^\pm 1 \text{ for all } j \in U(o(x)),$$

where $o(x)$ denotes the order of $x$, $U(n) := \{j : 1 \leq j \leq n, \gcd(j, n) = 1\}$, and $y \sim_G z$ denotes $y$ is conjugate to $z$ in $G$.

RS-subgroup

A subgroup $A$ of $G$ is an *RS-subgroup* of $G$, if every torsion element of $A$ is an RS-element in $G$.

Theorem ([DMS05], Theorems 8 & 9)

*If $A$ is a normal subgroup of a finite group $G$, then $A$ is an RS-subgroup of $G$ if, and only if, $\mathbb{Z}(U(\mathbb{Z}[G])) \cap \mathbb{Z}[A]$ consists of trivial units.*
Arbitrary cut groups and the $\mathcal{RS}$ property.

$\Phi(G)$: the FC-subgroup of $G$, i.e., the subgroup consisting of those elements of $G$ which have only finitely many conjugates in $G$.

$\Phi^+(G)$: torsion subgroup of $\Phi(G)$.

**Theorem ([DMS05], Main theorem)**

An arbitrary group $G$ is a cut-group, if and only if, every finite normal subgroup $A$ of $G$ satisfies (RS) in $G$:

$$\forall a \in A \text{ and } j \in \mathbb{Z}, \text{ with } (j, |a|) = 1, \text{ we have that } a^j \sim a \pm 1.$$ 

**Theorem ([DMS05], Main theorem, restating)**

A group $G$ is a cut-group if, and only if, $\Phi^+(G)$ is an $\mathcal{RS}$-subgroup of $G$. 
Arbitrary cut groups and the $RS$ property.

- $\Phi(G)$: the FC-subgroup of $G$, i.e., the subgroup consisting of those elements of $G$ which have only finitely many conjugates in $G$
- $\Phi^+(G)$: torsion subgroup of $\Phi(G)$.
Arbitrary cut groups and the $RS$ property.

- $\Phi(G)$: the FC-subgroup of $G$, i.e., the subgroup consisting of those elements of $G$ which have only finitely many conjugates in $G$
- $\Phi^+(G)$: torsion subgroup of $\Phi(G)$.

**Theorem ([DMS05], Main theorem)**

An arbitrary group $G$ is a cut-group, if and only if, every finite normal subgroup $A$ of $G$ satisfies $(RS)$ in $G$:

$\forall a \in A$ and $j \in \mathbb{Z}$, with $(j,|a|) = 1$, we have that, $a^j \sim a^{\pm 1}$. $(RS)$
Arbitrary cut groups and the $RS$ property.

- $\Phi(G)$: the FC-subgroup of $G$, i.e., the subgroup consisting of those elements of $G$ which have only finitely many conjugates in $G$
- $\Phi^+(G)$: torsion subgroup of $\Phi(G)$.

**Theorem ([DMS05], Main theorem)**

An arbitrary group $G$ is a cut-group, if and only if, every finite normal subgroup $A$ of $G$ satisfies $(RS)$ in $G$:

$$\forall \ a \in A \text{ and } j \in \mathbb{Z}, \text{ with } (j,|a|) = 1, \text{ we have that } , \ a^j \sim a^\pm 1. \ (RS)$$

**Theorem ([DMS05], Main theorem, restating)**

A group $G$ is a cut-group if, and only if, $\Phi^+(G)$ is an $RS$-subgroup of $G$. 

Lemma ([DMS05], Lemma 10)

If $G$ is an arbitrary group and $A$ a finite normal subgroup of $G$, then there exists a finite extension $H$ of $A$ such that $Z(U(Z[G]))) \cap Z[A] = Z(U(Z[H]))) \cap Z[A]$.

Proof. Define $\phi: G \to \text{Aut}(A)$, given by $g \mapsto \sigma_g$, where $\sigma_g: A \to A$ is given by $\sigma_g(a) = a g$, so that $\ker(\phi) = C := Cen_G(A)$.

Note that $|G/C| < \infty$, as $\langle 1 \rangle \to \ker(\phi) \to G \to \text{Im}(\phi)(\subseteq \text{Aut}(A)) \to \langle 1 \rangle$.

Define $H := A \rtimes \phi G/C$, with $\phi: G/C \to \text{Aut}(A)$, $g := gC \mapsto \phi(g) = \sigma_g$. $A$ is an RS-subgroup of $H$ if, and only if, $A$ is an RS-subgroup of $G$. 

S.Maheshwary (IISERM) Groups, rings and group rings June 10, 2019 8 / 31
Lemma ([DMS05], Lemma 10)

If $G$ is an arbitrary group and $A$ a finite normal subgroup of $G$, then there exists a finite extension $H$ of $A$ such that

$$\mathcal{Z}(U(\mathbb{Z}[G])) \cap \mathbb{Z}[A] = \mathcal{Z}(U(\mathbb{Z}[H])) \cap \mathbb{Z}[A].$$
Lemma ([DMS05], Lemma 10)

If $G$ is an arbitrary group and $A$ a finite normal subgroup of $G$, then there exists a finite extension $H$ of $A$ such that

$$
\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) \cap \mathbb{Z}[A] = \mathcal{Z}(\mathcal{U}(\mathbb{Z}[H])) \cap \mathbb{Z}[A].
$$

Proof.

Define $\phi : G \to \text{Aut}(A)$, given by $g \to \sigma_g$, where $\sigma_g : A \to A$ is given by $\sigma_g(a) = a^g$, with $g \text{ a group element.}$
Lemma ([DMS05], Lemma 10)

If \( G \) is an arbitrary group and \( A \) a finite normal subgroup of \( G \), then there exists a finite extension \( H \) of \( A \) such that

\[
\mathcal{Z}(U(\mathbb{Z}[G])) \cap \mathbb{Z}[A] = \mathcal{Z}(U(\mathbb{Z}[H])) \cap \mathbb{Z}[A].
\]

Proof.

Define \( \phi : G \to \text{Aut}(A) \), given by \( g \to \sigma_g \), where \( \sigma_g : A \to A \) is given by \( \sigma_g(a) = a^g \), so that \( \ker(\phi) = C := Cen_G(A) \).
Lemma ([DMS05], Lemma 10)

If $G$ is an arbitrary group and $A$ a finite normal subgroup of $G$, then there exists a finite extension $H$ of $A$ such that

$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) \cap \mathbb{Z}[A] = \mathcal{Z}(\mathcal{U}(\mathbb{Z}[H])) \cap \mathbb{Z}[A].$$

Proof.

Define $\phi : G \to \text{Aut}(A)$, given by $g \to \sigma_g$, where $\sigma_g : A \to A$ is given by $\sigma_g(a) = a^g$, so that $\ker(\phi) = C := \text{Cen}_G(A)$. Note that $|G/C| < \infty$, as

$$\langle 1 \rangle \to \ker(\phi) \to G \to \text{Im}(\phi) (\subseteq \text{Aut}(A)) \to \langle 1 \rangle.$$
Lemma ([DMS05], Lemma 10)

If $G$ is an arbitrary group and $A$ a finite normal subgroup of $G$, then there exists a finite extension $H$ of $A$ such that

$$Z(U(Z[G])) \cap Z[A] = Z(U(Z[H])) \cap Z[A].$$

Proof.

Define $\phi : G \to \text{Aut}(A)$, given by $g \to \sigma_g$, where $\sigma_g : A \to A$ is given by $\sigma_g(a) = a^g$, so that $\ker(\phi) = C := \text{Cen}_G(A)$. Note that $|G/C| < \infty$, as

$$\langle 1 \rangle \to \ker(\phi) \to G \to \text{Im}(\phi)(\subseteq \text{Aut}(A)) \to \langle 1 \rangle.$$  

Define $H := A \rtimes_{\phi} G/C$, with $\bar{\phi} : G/C \to \text{Aut}(A)$, $\bar{g} := gC \to \phi(g) = \sigma_g$. 

\qed
On arbitrary cut groups.

**Lemma ([DMS05], Lemma 10)**

If $G$ is an arbitrary group and $A$ a finite normal subgroup of $G$, then there exists a finite extension $H$ of $A$ such that

$$Z(U(Z[G])) \cap Z[A] = Z(U(Z[H])) \cap Z[A].$$

**Proof.**

Define $\phi : G \to \text{Aut}(A)$, given by $g \to \sigma_g$, where $\sigma_g : A \to A$ is given by $\sigma_g(a) = a^g$, so that $\ker(\phi) = C := Cen_G(A)$. Note that $|G/C| < \infty$, as

$$\langle 1 \rangle \to \ker(\phi) \to G \to \text{Im}(\phi)(\subseteq \text{Aut}(A)) \to \langle 1 \rangle.$$

Define $H := A \rtimes \frac{G}{C}$, with $\bar{\phi} : G/C \to \text{Aut}(A)$, $\bar{g} := gC \to \phi(g) = \sigma_g$.

$A$ is an RS-subgroup of $H$ if, and only if, $A$ is an RS-subgroup of $G$. 

S.Maheshwary (IISERM) Groups, rings and group rings June 10, 2019 8 / 31
An arbitrary cut group

Theorem ([BMP19], Theorem 2)

For every group $G$, the following statements are equivalent:

(i) $G$ is a cut-group.

(ii) $\Phi^+(G)$ is an RS-subgroup of $G$.

(iii) $\mathbb{Z}(\mathcal{U}(\mathbb{Z}[G])) \cap \mathbb{Z}[\Phi^+(G)] = \pm T(\mathbb{Z}(G))$, where $T(\mathbb{Z}(G))$ denotes the torsion subgroup of $\mathbb{Z}(G)$.

(iv) $\pm G = \mathcal{N}_\mathcal{U}(G)$, the normalizer of $G$ in $\mathcal{U} := \mathcal{U}(\mathbb{Z}[G])$.

Remark

- Every group with $\Phi^+(G) = \{1\}$ is a cut group.
- In particular, any torsion free group is a cut group.
An arbitrary cut group

- Clearly, the property of being an RS-subgroup is subgroup-closed. (A subgroup of a finite cut group may not be a cut group.)

- This property also turns out to be closed under taking quotients by finite normal subgroups. (If $G$ is a finite cut-group, then so is $G/N$ for every $N \trianglelefteq G$ [RS90]. As a generalization of this fact, we have the following:)

**Theorem ([BMP19], Theorem 4)**

*If $G$ is a cut-group, then $G/N$ is a cut-group for every finite normal subgroup $N$ of $G$.*

**Proof.**

This follows by checking that if $A$ is an RS subgroup of $G$, containing $N$, then $A/N$ is an RS subgroup of $G/N$. \qed
Theorem ([BMP19], Theorem 5)

(i) Let $G$ be a normal subgroup of the group $\Pi$ and $Q = \Pi/G$.

$$(1 \to G \to \Pi \to Q = \Pi/G \to 1)$$

(a) If $Q$ is a cut-group, and $G \cap \Phi^+(\Pi) = \{1\}$ (in particular, if $\Phi^+(G') = \{1\}$), then $\Pi$ is a cut-group.

(b) If $\Phi^+(Q) = \{1\}$, and $\Phi^+(G)$ is an RS-subgroup of $\Pi$ (in particular, if $G$ is a cut-group), then $\Pi$ is a cut-group.

(ii) If $\Pi = G \ast_A G'$ is an amalgam of arbitrary groups $G$ and $G'$ with the amalgamated subgroup $A$ an RS-subgroup of $G$ or $G'$, then $\Pi$ is a cut-group, provided $A \neq G$ and $A \neq G'$. In particular, the free product of arbitrary non-trivial groups is a cut-group.

(iii) If $\Pi$ is an HNN extension of a group $G$ over isomorphic subgroups $A$ and $B$ such that one of $A$ or $B$ is an RS-subgroup of $G$, then $\Pi$ is a cut-group.
Cut groups under extensions

Examples

1. Not every finite simple group is a cut-group (a complete list of finite cut groups can be found in [BCJM]). But, if \( G \) is an infinite simple group, then clearly \( \Phi^+(G) = \{1\} \) and therefore \( G \) is a cut-group. Furthermore (i)(a) of above theorem, implies that Extension of an infinite simple group by a cut-group is a cut group.

2. Recall that \( \text{PSL}(n,k) \) is simple if \( k \) is a field of characteristic 0 and \( n \geq 2 \). If the roots of unity in \( k \) are of exponent dividing 4 or 6 (e.g., if \( k = \mathbb{Q} \) or \( \mathbb{R} \)), then above theorem ((i)(b)) yields that \( \text{SL}(n,k) \) is a cut-group for \( n \geq 2 \).
Cut groups under extensions

Examples

1. Not every finite simple group is a cut-group (a complete list of finite cut groups can be found in [BCJM]). But, if $G$ is an infinite simple group, then clearly $\Phi^+(G) = \{1\}$ and therefore $G$ is a cut-group. Furthermore (i)(a) of above theorem, implies that...
Cut groups under extensions

Examples

1. Not every finite simple group is a cut-group (a complete list of finite cut groups can be found in [BCJM]). But, if $G$ is an infinite simple group, then clearly $\Phi^+(G) = \{1\}$ and therefore $G$ is a cut-group. Furthermore (i)(a) of above theorem, implies that

\textit{Extension of an infinite simple group by a cut-group is a cut group.}
Cut groups under extensions

Examples

1. Not every finite simple group is a cut-group (a complete list of finite cut groups can be found in [BCJM]). But, if $G$ is an infinite simple group, then clearly $\Phi^+(G) = \{1\}$ and therefore $G$ is a cut-group. Furthermore (i)(a) of above theorem, implies that

   *Extension of an infinite simple group by a cut-group is a cut group.*

2. Recall that $PSL(n,k)$ is simple if $k$ is a field of characteristic 0 and $n \geq 2$. If the roots of unity in $k$ are of exponent dividing 4 or 6 (e.g., if $k = \mathbb{Q}$ or $\mathbb{R}$), then above theorem ((i)(b)) yields that
Cut groups under extensions

Examples

1. Not every finite simple group is a cut-group (a complete list of finite cut groups can be found in [BCJM]). But, if $G$ is an infinite simple group, then clearly $\Phi^+(G) = \{1\}$ and therefore $G$ is a cut-group. Furthermore (i)(a) of above theorem, implies that

   Extension of an infinite simple group by a cut-group is a cut group.

2. Recall that $PSL(n,k)$ is simple if $k$ is a field of characteristic 0 and $n \geq 2$. If the roots of unity in $k$ are of exponent dividing 4 or 6 (e.g., if $k = \mathbb{Q}$ or $\mathbb{R}$), then above theorem ((i)(b)) yields that

   $SL(n,k)$ is a cut-group for $n \geq 2$. 
Cut groups under extensions

Examples

3. It is known that the modular group $\text{PSL}(2, \mathbb{Z})$ is the free product of cyclic groups $C_2$ and $C_3$. Thus, by (ii) above, we get that the modular group $\text{PSL}(2, \mathbb{Z})$ is a cut-group.

4. Observe that $\text{SL}(2, \mathbb{Z})$ is isomorphic to $C_4 \ast C_2 \ast C_6$ and thus, by (ii) in above theorem implies that, $\text{SL}(2, \mathbb{Z})$ is a cut-group.

5. The Baumslang Solitar group $\text{BS}(m,n) := \langle a, t \mid t^{-1}a^mt = a^n \rangle$, where $m$ and $n$ are non-zero integers, is an HNN extension. Thus, by (iii) of above result, we obtain that the Baumslag Solitar groups $\text{BS}(m,n)$ are cut-groups.
3. It is known that the modular group $PSL(2, \mathbb{Z})$ is the free product of cyclic groups $C_2$ and $C_3$. Thus, by (ii) above, we get that
3. It is known that the modular group $PSL(2, \mathbb{Z})$ is the free product of cyclic groups $C_2$ and $C_3$. Thus, by (ii) above, we get that

The modular group $PSL(2, \mathbb{Z})$ is a cut-group.
3. It is known that the modular group $PSL(2, \mathbb{Z})$ is the free product of cyclic groups $C_2$ and $C_3$. Thus, by (ii) above, we get that

The modular group $PSL(2, \mathbb{Z})$ is a cut-group.

4. Observe that $SL(2, \mathbb{Z})$ is isomorphic to $C_4 \ast_{C_2} C_6$ and thus, by (ii) in above theorem implies that,
Cut groups under extensions

Examples

3. It is known that the modular group $PSL(2, \mathbb{Z})$ is the free product of cyclic groups $C_2$ and $C_3$. Thus, by (ii) above, we get that

$$\text{The modular group } PSL(2, \mathbb{Z}) \text{ is a cut-group.}$$

4. Observe that $SL(2, \mathbb{Z})$ is isomorphic to $C_4 \ast C_2 C_6$ and thus, by (ii) in above theorem implies that,

$$SL(2, \mathbb{Z}) \text{ is a cut-group.}$$
3. It is known that the modular group $PSL(2, \mathbb{Z})$ is the free product of cyclic groups $C_2$ and $C_3$. Thus, by (ii) above, we get that

The modular group $PSL(2, \mathbb{Z})$ is a cut-group.

4. Observe that $SL(2, \mathbb{Z})$ is isomorphic to $C_4 \ast C_2 C_6$ and thus, by (ii) in above theorem implies that,

$SL(2, \mathbb{Z})$ is a cut-group.

5. The Baumslang Solitar group $BS(m, n) := \langle a, t \mid t^{-1}a^mt = a^n \rangle$, where $m$ and $n$ are non-zero integers, is an HNN extension. Thus, by (iii) of above result, we obtain that
Cut groups under extensions

Examples

3. It is known that the modular group $PSL(2, \mathbb{Z})$ is the free product of cyclic groups $C_2$ and $C_3$. Thus, by (ii) above, we get that

The modular group $PSL(2, \mathbb{Z})$ is a cut-group.

4. Observe that $SL(2, \mathbb{Z})$ is isomorphic to $C_4 \ast C_2 C_6$ and thus, by (ii) in above theorem implies that,

$SL(2, \mathbb{Z})$ is a cut-group.

5. The Baumslang Solitar group $BS(m, n) := \langle a, t \mid t^{-1}a^mt = a^n \rangle$, where $m$ and $n$ are non-zero integers, is an HNN extension. Thus, by (iii) of above result, we obtain that

The Baumslag Solitar groups $BS(m, n)$ are cut-groups.
Theorem (Finite case, ([BMP17], Theorem 5))

A finite non-abelian metacyclic group is a cut-group if, and only if, it is isomorphic to one of the following groups:

\[
\langle a, b \mid a^n = 1, b^t = 1, b^{-1}ab = a^{n-1} \rangle, \quad t = 2, 4, 6, \quad n = 3, 4, 6;
\]
\[
\langle a, b \mid a^4 = 1, b^t = a^2, b^{-1}ab = a^3 \rangle, \quad t = 2, 4, 6;
\]
\[
\langle a, b \mid a^6 = 1, b^2 = a^3, b^{-1}ab = a^5 \rangle;
\]
\[
\langle a, b \mid a^n = 1, b^{\varphi(n)} = 1, b^{-1}ab = a^{\lambda n} \rangle, \quad n = 5, 7, 9, 10, 14, 18;
\]
\[
\langle a, b \mid a^n = 1, b^{\varphi(n)/j} = 1, b^{-1}ab = a^{\lambda n} \rangle, \quad j = 1, 2, \quad n = 7, 9, 14, 18;
\]
\[
\langle a, b \mid a^8 = 1, b^t = 1, b^{-1}ab = a^r \rangle, \quad t = 2, 4, \quad r = 3, 5;
\]
\[
\langle a, b \mid a^{12} = 1, b^t = 1, b^{-1}ab = a^5 \rangle, \quad t = 2, 4;
\]
\[
\langle a, b \mid a^{12} = 1, b^t = a^\ell, b^{-1}ab = a^7 \rangle, \quad t = 2, 6, \quad \ell = t, 12;
\]
\[
\langle a, b \mid a^{15} = 1, b^4 = 1, b^{-1}ab = a^2 \rangle;
\]
\[
\langle a, b \mid a^{16} = 1, b^4 = 1, b^{-1}ab = a^r \rangle, \quad r = 3, 5;
\]
\[
\langle a, b \mid a^{20} = 1, b^4 = 1, b^{-1}ab = a^r \rangle, \quad r = 3, 13;
\]
\[
\langle a, b \mid a^{20} = 1, b^4 = a^{10}, b^{-1}ab = a^3 \rangle;
\]
\[
\langle a, b \mid a^{21} = 1, b^6 = 1, b^{-1}ab = a^r \rangle, \quad r = 2, 10;
\]
\[
\langle a, b \mid a^{28} = 1, b^6 = a^\ell, b^{-1}ab = a^{11} \rangle, \quad \ell = 14, 28;
\]
\[
\langle a, b \mid a^{30} = 1, b^4 = 1, b^{-1}ab = a^{17} \rangle;
\]
\[
\langle a, b \mid a^{36} = 1, b^6 = a^\ell, b^{-1}ab = a^{7} \rangle, \quad \ell = 6, 36;
\]
\[
\langle a, b \mid a^{42} = 1, b^6 = 1, b^{-1}ab = a^r \rangle, \quad r = 11, 19;
\]
Theorem (Infinite case, ([BMP19], Theorem 6))

An infinite non-abelian metacyclic group is a cut-group if, and only if, it is isomorphic to one of the following groups:

(i) \( \langle a, b \mid b^n = 1, ba = a^{-1}b \rangle, \ n \in \{0, 2, 4, 6, 8, 12\} \);

(ii) \( \langle a, b \mid a^m = 1, ba = a^r b \rangle, \ m \geq 3, \ 1 \neq r \in U(m) \) and \( U(m) = \langle -1, r \rangle \).
Theorem (Finite, ([Mah18], Theorem 3))

A finite nilpotent group $G$ has the cut-property if, and only if, $G$ is one of the following:

(i) a 2-group such that for all $x \in G$, $x^3 \sim x$ or $x^{-1}$;

(ii) a 3-group such that for all $x \in G$, $x^2 \sim x^{-1}$;

(iii) a direct sum $H \oplus K$, a real group $H$ and a group $K(\neq \langle 1 \rangle)$ satisfying (i) & (ii) respectively.

In particular $\pi(G) \subseteq \{2, 3\}$.
Theorem (Finite, ([Mah18], Theorem 3))

A finite nilpotent group $G$ has the cut-property if, and only if, $G$ is one of the following:

(i) a 2-group such that for all $x \in G$, $x^3 \sim x$ or $x^{-1}$;

(ii) a 3-group such that for all $x \in G$, $x^2 \sim x^{-1}$;

(iii) a direct sum $H \oplus K$, a real group $H$ and a group $K(\neq \langle 1 \rangle)$ satisfying (i) & (ii) respectively.

In particular $\pi(G) \subseteq \{2, 3\}$.

Theorem ([BMP19], Corollary 8)

Let $A$ be a normal RS-subgroup of a finite nilpotent group $G$, then $\pi(A) \subseteq \{2, 3\}$. 
Theorem (Infinite, [BMP19], Theorem 10)

A nilpotent group \( G \) is a cut-group if, and only if, either \( \Phi^+(G) = \{1\} \) or \( \Phi^+(G) \) is a \( \{2, 3\} \)-group and the following conditions hold:

(i) for all \( a \in \Phi^+(G)_2 \), \( a^3 \sim_G a^{\pm 1} \);

(ii) for all \( a \in \Phi^+(G)_3 \), \( a^2 \sim_G a^{-1} \),

where \( G_p \) denotes the subset of \( G \) consisting of \( p \)-elements in \( G \).
A finite $p$-group $G$ is a cut-group if, and only if, $G$ satisfies one of the following:

(i) $p = 2$ and for all $x \in G$, $x^3 \sim x$ or $x^{-1}$;

(ii) $p = 3$ and for all $x \in G$, $x^2 \sim x^{-1}$;
$p$-groups and the RS-property

**Theorem (Finite case)**

A finite $p$-group $G$ is a cut-group if, and only if, $G$ satisfies one of the following:

(i) $p = 2$ and for all $x \in G$, $x^3 \sim x$ or $x^{-1}$;
(ii) $p = 3$ and for all $x \in G$, $x^2 \sim x^{-1}$;

**Theorem**

A non-trivial normal subgroup $A$ of a finite $p$-group $G$ is an RS-subgroup of $G$ if, and only if, one of the following holds:

(i) $p = 2$ and $a^3 \sim_G a^{\pm 1}$ for all $a \in G$;
(ii) $p = 3$ and $a^2 \sim_G a^{-1}$ for all $a \in G$. 
Theorem (Infinite, [BMP19], Theorem 9)

A $p$-group $G$ is a cut-group if, and only if, one of the following holds:

(i) $p = 2$ and $a^3 \sim_G a \pm 1$ for all $a \in \Phi^+(G)$;

(ii) $p = 3$ and $a^2 \sim_G a - 1$ for all $a \in \Phi^+(G)$;

(iii) $\Phi^+(G) = \{1\}$.

Remark

It may be noted that, while a finite $p$-group, which is a cut-group, must necessarily be a 2-group or a 3-group, this is not the case for an infinite $p$-group to be a cut-group. For example, for any prime $p$, the wreath product $C_p \wr A$, where $A$ is a direct sum of infinitely many copies of $C_p$, is a cut-group, since $\Phi(C_p \wr A) = \{1\}$.
Theorem (Infinite, [BMP19], Theorem 9)

A $p$-group $G$ is a cut-group if, and only if, one of the following holds:

(i) $p = 2$ and $a^3 \sim_G a^{\pm 1}$ for all $a \in \Phi^+(G)$;
(ii) $p = 3$ and $a^2 \sim_G a^{-1}$ for all $a \in \Phi^+(G)$;
(iii) $\Phi^+(G) = \{1\}$.

Remark: It may be noted that, while a finite $p$-group, which is a cut-group, must necessarily be a 2-group or a 3-group, this is not the case for an infinite $p$-group to be a cut-group. For example, for any prime $p$, the wreath product $C_p \wr A$, where $A$ is a direct sum of infinitely many copies of $C_p$, is a cut-group, since $\Phi(C_p \wr A) = \{1\}$. 
Arbitrary $p$-cut-groups

**Theorem (Infinite, [BMP19], Theorem 9)**

A $p$-group $G$ is a cut-group if, and only if, one of the following holds:

(i) $p = 2$ and $a^3 \sim_G a^{\pm 1}$ for all $a \in \Phi^+(G)$;
(ii) $p = 3$ and $a^2 \sim_G a^{-1}$ for all $a \in \Phi^+(G)$;
(iii) $\Phi^+(G) = \{1\}$.

**Remark**

It may be noted that, while a finite $p$-group, which is a cut-group, must necessarily be a 2-group or a 3-group, this is not the case for an infinite $p$-group to be a cut-group. For example, for any prime $p$, the wreath product $C_p \wr A$, where $A$ is a direct sum of infinitely many copies of $C_p$, is a cut-group, since $\Phi(C_p \wr A) = \{1\}$. 
Theorem

Given a finite group $G$, the following statements are equivalent:

(i) $\mathbb{Z}(U(\mathbb{Z}G))$ is trivial.

(ii) $\rho(G) = 0$, where $\rho(G)$ denotes the rank of $\mathbb{Z}(U(\mathbb{Z}G))$ ($\mathbb{Z}(U(\mathbb{Z}G)) = \pm \mathbb{Z}(G) \times F$).
On the torsion-free ranks

**Theorem**

Given a finite group $G$, the following statements are equivalent:

(i) $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is trivial.

(ii) $\rho(G) = 0$, where $\rho(G)$ denotes the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ ($\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) = \pm \mathbb{Z}(G) \times F$).

**Theorem ([BMP19], Theorem 11)**

Given a normal subgroup $A$ of a finite group $G$, the following statements are equivalent:

(i) $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \cap (1 + \Delta(G)\Delta(A))$ is trivial.

(ii) $\rho(G) = \rho(G/A) = \text{the torsion free rank of } \pi(\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))),$ where $\rho(G)$ denotes the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ and $\pi: \mathbb{Z}G \to \mathbb{Z}[G/A]$ is the ring homomorphism induced by the natural projection $G \to G/A$. 
Theorem ([Mah18], [Bäc18])

Let $G$ be a finite solvable group such that every element of $G$ has prime power order. Then, $G$ has the cut-property if, and only if, every element $x \in G$ satisfies one of the following conditions:

(i) $o(x) = 2^a$, $a \geq 0$ and $x^3 \sim x$ or $x^{-1}$;
(ii) $o(x) = 7$ or $3^b$, $b \geq 1$ and $x^5 \sim x^{-1}$;
(iii) $o(x) = 5$ and $x^3 \sim x^{-1}$.

Let $G$ be a solvable cut group. Then $\pi(G) \subseteq \{2, 3, 5, 7\}$.
Theorem ([Mah18], [Bäc18])

Let $G$ be a finite solvable group such that every element of $G$ has prime power order. Then, $G$ has the cut-property if, and only if, every element $x \in G$ satisfies one of the following conditions:

(i) $o(x) = 2^a$, $a \geq 0$ and $x^3 \sim x$ or $x^{-1}$;
(ii) $o(x) = 7$ or $3^b$, $b \geq 1$ and $x^5 \sim x^{-1}$;
(iii) $o(x) = 5$ and $x^3 \sim x^{-1}$.

Let $G$ be a solvable cut group. Then $\pi(G) \subseteq \{2,3,5,7\}$.

Theorem ([BMP19], Corollary 12)

Let $A$ be a finite normal subgroup of a solvable group $G$; $H := A \rtimes G/C_G(A)$. If

(i) $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[H])) \cap (1 + \Delta(H)\Delta(A))$ is trivial;
(ii) $G/C_G(A)$ is a cut-group.

Then, $\pi(A) \subseteq \{2,3,5,7\}$.
Symmetric central units

Theorem ([Seh93], [JdR15])

If $G$ is a finite abelian group, then $U(Z[G])$ is the direct product of $\pm G$ and a torsion free group of symmetric units. (A unit $u = \sum u_g g \in Z[G]$, is called symmetric, if $u = u^\ast := \sum u_g g - 1$.)

Theorem

For every group $G$, the following statements hold:

(i) There is an exact sequence $1 \to \pm Z(G) \to Z(U(Z[G])) \to S_0 \to 1$, where $S_0$ is a torsion free subgroup of $S := Z(S)$ consisting of symmetric central units.

(ii) If $u \in Z(U(Z[G]))$, then there exists an element $g \in Z(G)$ such that $u = gu^\ast$. Furthermore, $u^2 \in Z(G)S$.

(iii) If the symmetric central units of $Z[G]$ are trivial, then so are all central units, i.e., $G$ is a cut-group.
Symmetric central units

Theorem ([Seh93],[JdR15])

If $G$ is a finite abelian group, then $U(\mathbb{Z}[G])$ is the direct product of $\pm G$ and a torsion free group of symmetric units. (A unit $u = \sum u_g g \in \mathbb{Z}[G]$, is called symmetric, if $u = u^* := \sum u_g g^{-1}$.)
Symmetric central units

Theorem ([Seh93],[JdR15])

If $G$ is a finite abelian group, then $\mathcal{U}(\mathbb{Z}[G])$ is the direct product of $\pm G$ and a torsion free group of symmetric units. (A unit $u = \sum u_g g \in \mathbb{Z}[G]$, is called symmetric, if $u = u^* := \sum u_g g^{-1}$.)

Theorem

For every group $G$, the following statements hold:

(i) There is an exact sequence $1 \rightarrow \pm \mathbb{Z}(G) \rightarrow \mathbb{Z}(\mathcal{U}(\mathbb{Z}[G])) \rightarrow S_0 \rightarrow 1$, where $S_0$ is a torsion free subgroup of $S := \mathbb{Z}_S(\mathcal{U}(\mathbb{Z}[G]))$, consisting of symmetric central units.

(ii) If $u \in \mathbb{Z}(\mathcal{U}(\mathbb{Z}[G]))$, then there exists an element $g \in \mathbb{Z}(G)$ such that $u = gu^*$. Furthermore, $u^2 \in \mathbb{Z}(G)S$.

(iii) If the symmetric central units of $\mathbb{Z}[G]$ are trivial, then so are all central units, i.e., $G$ is a cut-group.


Group discussions

\[ \mathbb{Q}G \text{ says to } \mathbb{Z}G \]

Over the years, has situation arose, everyone is interested, how I decompose? Thank \( \mathbb{Q} \), for being rational, making me semisimple.

My situation is even more ideal, when my \( \mathbb{G} \) is monomial. For then, to my rescue, comes Shoda pair giving information, quite fair.

Hey \( \mathbb{Z} \mathbb{G} \), mine is field, yours is ring, Are your properties, still interesting? For, I have a lot to offer, whereas, you have nothing in order.
Over the years, has situation arose, every one is interested, how I decompose?

Thank $\mathbb{Q}$, for being rational, making me semisimple.
QG says to ZG

Over the years, has situation arose, every one is interested, how I decompose?
Thank ℚ, for being rational, making me semisimple.

My situation is even more ideal, when my G is monomial.
For then, to my rescue, comes Shoda pair giving information, quite fair.
Over the years, has situation arose, every one is interested, how I decompose?

Thank \( \mathbb{Q} \), for being rational, making me semisimple.

My situation is even more ideal, when my \( G \) is monomial.

For then, to my rescue, comes Shoda pair giving information, quite fair.

Hey \( \mathbb{Z}G \), mine is field, yours is ring, Are your properties, still interesting?

For, I have a lot to offer, whereas, you have nothing in order.
$\mathbb{Z}G$ replies to $\mathbb{Q}G$

Hey $\mathbb{Q}G$, do you forget?

I do not have to regret,
as I am order in you,
with lots of articles in queue.

My ring is integral
for my units, even if central;
the interest not really fades,
even when worked for many decades.

I have a long history,
and my units are still a mystery.
Hey \( \mathbb{Q}G \), do you forget?

I do not have to regret, as I am order in you, with lots of articles in queue.
Hey $\mathbb{Q}G$, do you forget?

I do not have to regret,
as I am order in you,
with lots of articles in queue.

My ring is integral
for my units, even if central;
the interest not really fades,
even when worked for many decades.
Hey $\mathbb{Q}G$, do you forget?

I do not have to regret,
as I am order in you,
with lots of articles in queue.

My ring is integral
for my units, even if central;
the interest not really fades,
even when worked for many decades.

I have a long history,
and my units are still a mystery.
Group discussions (continued…)

$\mathbb{Q}G$ to $\mathbb{Z}G$

Ring integral, units central, what do you do with rhymes, when all are just trivial at times!

And, if that’s not the case, unit groups are even harder to chase.

For unit groups in orders, though comensurable, are generally huge, and hardly measurable.

S. Maheshwary (IISERM)
Group discussions (continued...)

QG to ZG

Ring integral,
units central,
what do you do with rhymes,
when all are just trivial at times!
Ring integral, units central, 
what do you do with rhymes, when all are just trivial at times!

And, if that's not the case, unit groups are even harder to chase. For unit groups in orders, though comensurable, are generally huge, and hardly measurable.
Group discussions (continued…)

\[ \mathbb{Z}(U(\mathbb{Z}G)) \text{ to } \mathbb{Q}G, \text{ again} \]

Yes, lots of elements, only few torsion, I generally contain, a group free abelian. But my free rank is finite, if so is my group underlying. And situation is even more cordial, if all my elements are trivial. In that case, I tell you what, my group \( G \), is called CUT. \( G \) is also called inverse semi-rational, and has engaging properties, both local and global.
Yes, lots of elements, only few torsion, I generally contain, a group free abelian. But my free rank is finite, if so is my group underlying.
Yes, lots of elements, only few torsion, 
I generally contain, a group free abelian. 
But my free rank is finite, 
if so is my group underlying.

And situation is even more cordial, 
if all my elements are trivial. 
In that case, I tell you what, 
my group $G$, is called CUT.
Group discussions (continued...)

$\mathcal{Z}(U(\mathbb{Z}G))$ to $\mathbb{Q}G$, again

Yes, lots of elements, only few torsion,
I generally contain, a group free abelian.
   But my free rank is finite,
      if so is my group underlying.

And situation is even more cordial,
if all my elements are trivial.
   In that case, I tell you what,
      my group $G$, is called CUT.

$G$ is also called inverse semi-rational,
and has engaging properties, both local and global.
Group discussions (continued...)

$\mathcal{Z}(U(\mathbb{Z}G))$ to $\mathbb{Q}G$, again
In this situation, my opponent, restriction is on your component. For, it’s centre can’t be arbitrary, just a field of rationals, or quadratic imaginary.
In this situation, my opponent, restriction is on your component. For, it’s centre can’t be arbitrary, just a field of rationals, or quadratic imaginary.

Information on central height of \( \mathcal{U}(\mathbb{Z}G) \), restriction on prime spectrum of \( G \), connections with normaliser property, what more proofs I give, of my non-triviality!
In this situation, my opponent, restriction is on your component. For, it’s centre can’t be arbitrary, just a field of rationals, or quadratic imaginary.

Information on central height of $\mathcal{U}(\mathbb{Z}G)$, restriction on prime spectrum of $G$, connections with normaliser property, what more proofs I give, of my non-triviality!

I guess by now you agree, I have a lot in store, the experts in group rings, need to explore!
THANK YOU!!!