Irrationality of quotient varieties

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Difficult problem: Which varieties are birational to $\mathbb{P}^n$ (= rational)?

$k(x) \cong k(t_1, \ldots, t_n)$

Problem: Given a nice (smooth projective) variety $X/k$,
is $X$ birational to some $\mathbb{P}^n$?
Difficult problem: Which varieties are birational to $\mathbb{P}^n$ \((=\text{rational})\)?

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Problem: Given a nice (smooth projective) variety \(X/k\), is \(X\) birational to some \(\mathbb{P}^n\)?

Lüroth problem: nice \(\equiv\) unirational

\[ \exists \mathbb{P}^n \overset{\text{dom.}}{\rightarrow} X \iff k(X) \hookrightarrow k(t_1, \ldots, t_n) \]

Lüroth problem has a positive answer over \(k\) when \(\dim X \leq 2\).
Noether problem: nice $\Rightarrow$ quotient variety

$$V/G = \text{Spec } k[V]^G$$

Vector space representation of $G$ $\Rightarrow$ finite group $\Rightarrow$ $G$-invariant regular functions

Example: $\mathbb{C}_2 \curvearrowright V = \mathbb{A}_k^2$ $\xrightarrow{(x,y) \mapsto (-x,-y)}$

$$V/\mathbb{C}_2 = \text{Spec } k[x^2, xy, y^2] \cong \text{Spec } k[u, v, w]/(uw-v^2)$$

$k(V/\mathbb{C}_2) \cong k(u, v)$

$\Rightarrow$ $V/\mathbb{C}_2 \sim \mathbb{P}^2$
Motivation:

If $V/G$ is birational to $\mathbb{P}^n$ over $k = \mathbb{Q}$, then one can strongly solve the inverse Galois problem for $G$.

\[ G \cong \text{Gal}(E/\mathbb{Q}) \text{ for some } E \]

(Noether, Hilbert ... )
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(Noether, Hilbert ...)  

Unfortunately, Noether problem does not have a positive answer in general, smallest counterexample is $C_8$.

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Assume from now on that $k=\mathbb{C}$. 

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  $G = \text{Sym}(n) \implies V/G$ rational

[\text{Example's Note}: $\mathcal{C}(V)^G = \text{ (symmetric polynomials) }$]
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- $G$ abelian $\Rightarrow$ $V/G$ rational \hspace{1cm} (Fischer 1915)

- $G = \text{Sym}(n)$ $\Rightarrow$ $V/G$ rational

  $\mathcal{I}(V)^G =$ (symmetric polynomials)

- $G$ 2-group & $|G| \leq 32$ $\Rightarrow$ $V/G$ rational

  (Chu & Hu & Kang & Prokhorov 2008)
To prove that a variety $X$ is not (stably) rational, one uses a birational invariant $I : \{ \text{varieties} \} \rightarrow \{ \text{abelian groups} \}$ with

$$I(X) \neq 0 \quad \& \quad I(\mathbb{P}^n) = 0.$$
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- $\text{Br}_{\text{nr}}(X) := \bigcap_{\substack{R \text{ D.V.R.} \\ K(R) = \mathbb{C}(X)}} \text{image} \left( \text{Br}(R) \rightarrow \text{Br}(\mathbb{C}(X)) \right)$ \hspace{1cm} (Saltman 1984 for $X = V/G$)

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  Unramified Brauer group

For $X$ smooth projective unirational:

$$\text{Br}_{\text{nr}}(X) = \text{Br}(X) = \text{Torsion } H^3_{\text{sing}}(X(\mathbb{C}), \mathbb{Z})$$

(Bogomolov 1989)
This invariant is computable for $X = V/G$:

$$\text{Br}_{nr}(V/G) = \bigcap_{A \leq G, A \text{ abelian}} \ker \left( H^2(G, \mathbb{C}^*) \xrightarrow{\text{res}} H^2(A, \mathbb{C}^*) \right)$$

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This invariant is computable for $X = V/G$:

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$$H^2_{nr}(V/G)$$

(Bogomolov 1989)

Bogomolov multiplier $B_0(G)$

Higher unramified cohomology (Colliot-Thélène & Ojanguren 1989)

$$H^i_{nr}(X) = E_2^{0,i} \text{ in Leray spectral sequence } (X_{\text{an}} \xrightarrow{id} X_{\text{zar}})$$

Only partial extensions of Bogomolov’s formula for $X = V/G$.

(Peyre 2008, Hoshi & Kang & Yamasaki 2018 computer calculations)
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- $B_0(\text{exponent } p \text{ class 2 quotient of } \pi_1(\Sigma_g) \text{ for } g \geq 2) \neq 0$

$$\prod_{i=1}^{g} [x_i, y_i] = 1$$ \hspace{1cm} (Saltman 1984)
Example:

- $B_0$ (abelian group) = 0
- $B_0$ (finite simple group) = 0 \hfill (Kunyavskii 2010)
- $B_0$ (exponent $p$ class 2 quotient of \( \pi_1(\Sigma_g) \) for $g \geq 2$) \( \neq 0 \)
  \[ \prod_{i=1}^{g} [x_i, y_i] = 1 \] \hfill (Saltman 1984)
- $B_0$ (some groups of order 64) \( \neq 0 \) \hfill (Chu, Hu & Kang & Kunyavskii 2012)
- $B_0$ (some groups of order $p^5$) \( \neq 0 \) \hfill (Hoshi, Kang & Kunyavskii 2012)
Quite surprisingly, $B_0$ is often non-trivial:

\[
\lim_{n \to \infty} \frac{\log \# \{ G : |G| = p^n, |B_0(G)| \geq M \}}{\log \# \{ G : |G| = p^n \}} = 1 \quad \forall M
\]

(With Sánchez 2018)

It follows that Noether problem has a strongly negative answer.
$B_0$ is an attractive object
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- **$K$-theory**

\[
B_0(G)^* \cong SK_1(Z_pG) = \text{kernel} \left( K_1(Z_pG) \longrightarrow K_1(Q_pG) \right)
\]

( Oliver 1980 )
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- **K-theory**

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  (Oliver 1980)

- **Representation theory**

  $$| \left( 1 + I_{\mathbb{F}_pG} \right)_{ab} | = p^{k(G)-1} \cdot | B_0(G) |$$

  \[ \# \text{Lin} \left( V(\mathbb{F}_pG) \right) \quad \# \text{Lin} \left( I_{\mathbb{F}_pG} \right) \]

  (with García-Rodríguez & Jaikin-Zapirain 2017)
Mathematical physics

Soft tensor braided autoequivalences of Drinfeld center of \( G \) category of compatible G-graded G-vector spaces

\[ \text{Out}_{2-\text{cl}}(G) \times B_0(G) \]

(Davydov 2014)
Mathematical physics

Soft tensor braided autoequivalences of Drinfeld center of $G$

category of compatible
G-graded G-vector spaces

$\text{Out}_{2-cl}(G) \times B_0(G)$

(Davydov 2014)

Moduli spaces

$\pi_0$ (Hurwitz space of connected curves $G$-covering $P^1$
with each type of local monodromies appearing sufficiently many times)

$\leftrightarrow B_0(G)$

(Ellenberg, Venkatesh & Westerland 2012)