Character degrees in $\pi$-separable groups

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π-separable groups and Hall π-subgroups

Let π be a set of primes. A number n is called a π-number if all its prime divisors are in π. Denote as π′ the complement set of π.

**Definition**

A group $G$ is called π-separable if, given a composition series

$$G = N_0 \triangleright N_1 \triangleright \ldots \triangleright N_r = \{1\},$$

then each factor group $N_1/N_{i+1}$ is either a π-group or a π′-group, i.e., its order is either a π-number or a π′-number.
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then each factor group $N_1/N_{i+1}$ is either a $\pi$-group or a $\pi'$-group, i.e., its order is either a $\pi$-number or a $\pi'$-number.

**Definition**

A subgroup $H \leq G$ is called Hall $\pi$-subgroup of $G$ if $|H| = |G|_{\pi}$, i.e., if its order is equal to the maximal product of powers of primes in $\pi$ which divides $|G|$.

If $G$ is a $\pi$-separable group, then it has a Hall $\pi$-subgroup and two distinct Hall $\pi$-subgroups are conjugated.
Character restriction and $B_{\pi}$-characters

Note that a $\{p\}$-separable group is a $p$-solvable group.

**Theorem (Isaacs, 1974)**

If $G$ is $p$-solvable, there exists a canonically defined set of characters $B_{p'}(G)$ such that the restriction to $p$-regular elements realizes a bijection between $B_{p'}(G)$ and $\text{IBr}_p(G)$.

Let $\chi^\pi$ be the restriction of a character $\chi$ to $\pi$-elements, i.e., to elements such that their order is a $\pi$-number. For any $\chi \in \text{Char}(G)$, we say that $\chi^\pi$ is a $\pi$-partial character. A $\pi$-partial character $\chi^\pi$ is irreducible if it cannot be written as a sum of two other $\pi$-partial characters.

**Theorem (Isaacs, 1982)**

If $G$ is $\pi$-separable, the set $I_{\pi}(G)$ of irreducible $\pi$-partial characters is a basis for the class functions on $\pi$-elements. Moreover, there exists a canonically defined set $B_{\pi}(G) \subseteq \text{Irr}(G)$ such that the restriction to $\pi$-elements realizes a bijection $B_{\pi}(G) \mapsto I_{\pi}(G)$. 
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Let $\chi^*$ be the restriction of a character $\chi$ to $\pi$-elements, i.e. to elements such that their order is a $\pi$-number. For any $\chi \in \text{Char}(G)$, we say that $\chi^*$ is a $\pi$-partial character. A $\pi$-partial character $\chi^*$ is irreducible if it cannot be written as a sum of two other $\pi$-partial characters.
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If $G$ is $\pi$-separable, the set $I_{\pi}(G)$ of irreducible $\pi$-partial characters is a basis for the class functions on $\pi$-elements.

Moreover, there exists a canonically defined set $B_{\pi}(G) \subseteq \text{Irr}(G)$ such that the restriction to $\pi$-elements realizes a bijection $B_{\pi}(G) \leftrightarrow I_{\pi}(G)$.
Variants of Ito-Michler theorem

A theorem of Michler affirms that a group $G$ has a normal Sylow $p$-subgroup if and only if $p$ does not divide the degree of any character in $\text{IBr}_p(G)$.

**Theorem (Isaacs, 2018)**

A $\pi$-separable group $G$ has a normal $\pi$-complement if and only if the degree of every character in $\text{B}_\pi(G)$ is a $\pi$-number.
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A $\pi$-separable group $G$ has a normal $\pi$-complement if and only if the degree of every character in $B_\pi(G)$ is a $\pi$-number.

On the other hand, the famous Theorem of Ito-Michler says that a group $G$ has a normal abelian Sylow $p$-subgroup if and only if $p$ does not divide the degree of any character in $\text{Irr}(G)$.

**Theorem**

Let $G$ be a finite $\pi$-separable group and let $p$ be any prime. Then, $G$ has a normal abelian Sylow $p$-subgroup if and only if $p \nmid \chi(1)$ for every $\chi \in B_\pi(G) \cup B_\pi'(G)$. 
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**Corollary**

If $G$ is a finite $\pi$-separable group, a prime $p$ divides the degree of some characters in $\text{Irr}(G)$ if and only if it divides the degree of some characters in $B_\pi(G) \cup B_\pi'(G)$.
Variants of Thompson’s theorem

Thompson’s theorem says that, if a prime $p$ divides the degree of each nonlinear character in $\text{Irr}(G)$, then $G$ has a normal $p$-complement. There are already variants, for ordinary characters, which involve more than one prime.

**Theorem (Navarro and Wolf, 2002)**

Let $G$ be a $\pi$-separable group and let $H$ be a Hall $\pi$-subgroup for $G$ and $N = N_G(H)$. Then, $\text{Irr}_{\pi'}(G) = \text{Lin}(G)$ if and only if $G' \cap N = H'$. 

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If we ask the same condition to hold for the characters in $B_\pi(G)$, or in $B_{\pi'}(G)$, we can have more detailed informations on the group structure.

**Theorem**

Let $G$ be a $\pi$-separable group, let $H$ be a Hall $\pi$-subgroup for $G$ and let $N = N_G(H)$. Then,

- $\text{Irr}_{\pi'}(G) \cap B_\pi(G) \subseteq \text{Lin}(G)$ if and only if $G' \cap H = H'$;
- $\text{Irr}_{\pi'}(G) \cap B_{\pi'}(G) \subseteq \text{Lin}(G)$ if and only if $G' \cap N \leq H$.
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If we ask the same condition to hold for the characters in $\text{B}_\pi(G)$, or in $\text{B}_{\pi'}(G)$, we can have more detailed informations on the group structure.

Theorem

Let $G$ be a $\pi$-separable group, let $H$ be a Hall $\pi$-subgroup for $G$ and let $N = N_G(H)$. Then,

- $\text{Irr}_{\pi'}(G) \cap \text{B}_\pi(G) \subseteq \text{Lin}(G)$ if and only if $G' \cap H = H'$;
- $\text{Irr}_{\pi'}(G) \cap \text{B}_{\pi'}(G) \subseteq \text{Lin}(G)$ if and only if $G' \cap N \leq H$;
- $\text{Irr}_{\pi'}(G) \cap (\text{B}_\pi(G) \cup \text{B}_{\pi'}(G)) \subseteq \text{Lin}(G)$ if and only if $\text{Irr}_{\pi'}(G) = \text{Lin}(G)$. 
Thank you!