Special Classes of Homogeneous Semilocal Rings

Corner Rings

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Basic definitions

- Let $J(R)$ be the Jacobson radical of $R$
- A ring $R$ is called **local** if $R/J(R)$ division ring
- A ring $R$ is called **homogeneous semilocal ring** if $R/J(R)$ simple artinian
- A ring $R$ is called **semiperfect ring** if $R/J(R)$ semisimple artinian and idempotents being liftable
- A ring $R$ is called **semilocal ring** if $R/J(R)$ semisimple artinian

$$\{\text{Local rings}\} \subset \{\text{Semiperfect rings}\}$$

$$\bigcap \quad \bigcap$$

$$\{\text{Homogeneous semilocal rings}\} \subset \{\text{Semilocal rings}\}$$
Family of Semilocal Rings

Semilocal rings

Homogeneous Semilocal rings

Local rings
Family of Semilocal Rings

- Semilocal rings
  - Semiperfect rings
    - Local rings
Maximal two-sided ideal

In non-commutative ring, local ring have a unique maximal left ideal (unique maximal right ideal) equivalent to have a unique maximal two-sided ideal but if the ring having a unique maximal two sided ideal is not equivalent to being local.

An extension class of local ring, which has a unique maximal two-sided ideal, is called homogeneous semilocal ring. The Jacobson radical of a homogeneous semilocal ring is its unique maximal two-sided ideal.
Homogeneous semilocal rings

A. Algebraic properties

Let $R$ be a homogeneous semilocal ring.

- The Jacobson radical $J(R)$ of $R$ is its unique maximal two-sided ideal, that is, every proper two-sided ideal of $R$ is contained in $J(R)$.
- $R$ has a unique simple module up to isomorphism.
- $R$ has only one indecomposable projective module up to isomorphism.
- All projective modules in $R$ are direct sums of copies of indecomposable projective module.
- Every homomorphic image of a homogeneous semilocal rings is homogeneous semilocal rings.
- Every maximal proper one-sided ideal contains a maximal proper two-sided ideal in $R$. 
Homogeneous semilocal rings

Algebraic properties

- If $S = R/J(R)$ is a homogeneous semilocal ring, then so is $R$.
- The Jacobson radical is the only primitive ideal.
- If $R$ is a homogenous semilocal ring, then $R$ is a Dedekind-finite.
- The center of a homogenous semilocal ring is commutative local ring
- A homogeneous semilocal ring has a nontrivial idempotents.

Example

Let $R = M_2(Z_4)$ it is evident that $R$ is a homogeneous semilocal ring and has a non trivial idempotent $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

- A homogeneous semilocal ring has no nontrivial idempotent if and only if it is local.
Homogeneous semilocal rings

Algebraic properties

**Proposition** [8, Proposition 2.1].

In a homogeneous semilocal ring $\mathcal{R}$ the Jacobson radical is the unique maximal proper two-sided ideal of $\mathcal{R}$, that is, contains all proper two-sided ideals of $\mathcal{R}$. Conversely, if a semilocal ring $\mathcal{R}$ has a unique maximal proper two-sided ideal, then $\mathcal{R}$ is homogeneous semilocal.

**Proposition 2.2.** [11, Proposition 2.2]. The following conditions are equivalent for a semilocal ring:

- $\mathcal{R}$ is a homogeneous semilocal ring;
- $\mathcal{R}$ has a unique maximal two-sided ideal;
- $\mathcal{R}$ has a unique right primitive ideal;
- $\mathcal{R}$ has a unique left primitive ideal.
Motivation

In noncommutative ring theory, the theory of idempotents plays an important role for studying the structure of related rings (such as corner and abelian rings). The corner rings are an important step to check the validity of Morita invariance of a ring.

An element \( e \) in a ring \( R \) is a full idempotent when \( e^2 = e \) and \( ReR = R \).

A ring property \( P \) is Morita invariant if and only if whenever a ring satisfies \( P \), then so does \( eRe \) for every full \( e \) and so does every matrix ring \( M_n(R) \) for every positive integer \( n \).

OR

for any ring \( R \) and full idempotent \( e \) in \( R \), \( R \) satisfies \( P \) if and only if the ring \( eRe \) satisfies \( P \).
Morita Invariant

Corisello and Facchini [2001] showed that the homogeneous semilocal property is transferred directly from the ring $R$ to the corner ring $eRe$ and to the matrix ring $M_n(R)$ for $n > 1$.

**Proposition**

1. If $R$ is a homogeneous semilocal ring and $n$ is a positive integer, then the ring $M_{n \times n}(R)$ of $n \times n$ matrices over $R$ is homogeneous semilocal ring. (the converse is true)

2. If $R$ is a homogeneous semilocal ring and $e$ is an idempotent element of $R$, then the corner ring $eRe$ is homogeneous semilocal ring. (the converse does not hold)

In spite of the local property is not Morita invariant, the homogeneous semilocal property is Morita invariant.
Unfortunately the Morita invariant is not preserved by homogeneous semilocal endomorphism ring like in the case of local and semilocal rings.

Example
Let $m, n, b$ be integers with $0 < m < n$ and $p$ prime. The endomorphism ring of the abelian group

$$\mathbb{Z}/p^m\mathbb{Z} \oplus \mathbb{Z}/p^n\mathbb{Z}$$

is semilocal, but not homogeneous semilocal.
Passing from $R$ to $eRe$ and Back

**Type (1)** There are many standard ring theoretic properties of $R$ are preserved by its corner ring $eRe$ such as left Artinian, left Noetherian, primitive, simple, and homogeneous semilocal are preserved by $eRe$ (and likewise to $(1-e)R(l-e)$) for every idempotent $e \in R$ [39, Lemma 2.7.12]; [11, Proposition 2.2 (ii)] but not the converse.

**Type (2)** Moreover, some standard ring theoretic properties such as semisimple Artinian, semilocal, and semiprimary hold in $R$ if and only if they hold both in $eRe$ and $(1-e)R(l-e)$, for every $e \in R$ [39, Proposition 2.7.14].
Passing from $R$ to $eRe$ and Back

**Type (3)** Some standard ring theoretic properties hold in $R$ if hold both in $eRe$ and $(1-e)R(l-e)$, for every nontrivial idempotent $e \in R$, like that clean.

**Type (4)** On the other hand, a corner ring satisfying some standard ring theoretic properties for which the basic ring do not satisfy those properties such as simple, homogeneous semilocal, right quasi duo, Dedekind finite, semicommutative, 2-primal, and NI.

**Type (5)** Furthermore, ring satisfy some standard ring theoretic properties for which the corner ring do not satisfy those properties such as, clean.
Example

Let $D$ be a division ring and by $R^N = M_N(D)$ the ring of all $N \times N$ matrices with the ring of all infinite matrices over $D$ with finite columns (each column contains only finitely many nonzero entries). Let $e$ be the idempotent in $R$ such that the $(1,1)$-entry of $e$ is $1_D$ and other entries of $e$ are $0_D$. Then $eRe(\cong D)$ is a homogeneous semilocal ring but $R$ is not.
Now we raise the following question:

Under what conditions, would a ring with homogeneous semilocal corner rings be a homogeneous semilocal ring?

Before answer this question, we quote the following useful result from [34].
**Lemma** [34, Lemma 5] Let $R$ be a ring and $e$ be a nonzero idempotent in $R$. If $M$ is the maximal ideal of $R$, then either $eMe = eRe$ or $eMe$ is a maximal ideal of $eRe$.

**Lemma** [34, Lemma 6] Let $R$ be a ring and $e$ be a nonzero idempotent in $R$. If $I$ is a maximal right (left) ideal of $R$, then either $eIe = eRe$ or $eIe$ is a maximal right (left) ideal of $eRe$.

**Proposition** 6 ([57], Proposition 2.13) . $R$ is semilocal if and only if $eRe$ and $(1 - e)R(1 - e)$ are both semilocal
Main Results

Actually, we consider some conditions under which the answer of our question may be affirmative.

**Theorem** Let $R$ be a ring and $M$ be a maximal ideal of $R$ satisfy $eMe \not\subseteq eRe$ for each a nonzero idempotent $e$ in $R$, then $R$ is a homogeneous semilocal ring if and only if $eRe$ and $(1-e)R(1-e)$ are both homogeneous semilocal.

**Corollary** Let $e^2 = e \neq 0$ be a proper idempotent in $R$. If $Re(eR)$ is a homogeneous semilocal ring then $eRe$ is a homogeneous semilocal ring.
Main Results

**Theorem** Let $R$ be a ring and $M$ be a maximal ideal of $R$ satisfy $eM e \subseteq eRe$ for each a nonzero idempotent $e$ in $R$, then the following are equivalent:

(i) $R$ is a homogeneous semilocal ring.
(ii) $eRe$ and $(1 - e)R(1 - e)$ are both homogeneous semilocal.
(iii) $R$ is a $W_n$-ring for some $n$ and has a unique maximal ideal.

**Definition** A ring $R$ is said to be $W_n$-ring if for any $r \in R$ there exists an integer $i$, $1 \leq i \leq n$, such that $1 - f_i(r)$ is invertible in $R$; where

$$f_1(r) = r, \quad f_i(r) = f_{i-1}(r)(1 - f_{i-1}(r))$$
Main Results

**Theorem** Let $R$ be a homogeneous semilocal ring such that the maximal ideal $M$ of $R$ is not equal zero (i.e. $M \neq 0$) but maximal ideal $M'$ of corner ring $eRe$ is equal zero (i.e. $M' = 0$) for every proper idempotent $e$ of $R$. Then for every proper idempotent $e \in R, eR(1-e) \subseteq M$ and $(1-e)Re \subseteq M$.

The following example shows that the condition “maximal ideal $M$ of $R$ is not equal zero” is necessary.

**Example** Let $R = M_n(F)$ be the all $n \times n$ matrices ring over a field $F$. It is clear that the maximal ideal $M$ of $R$ is equal zero. But $eR(1-e)$ and $(1-e)Re$ are non-zero for any proper idempotent $e \in R$. 
Theorem 2.2 [8]. Let $S$ be a finite normalizing extension of $R$. Then $S$ is artinian if and only if $R$ is artinian.

Theorem 2.3. Let $S$ be a finite normalizing extension of $R$. Then $S$ is a semilocal ring if and only if $R$ is a semilocal ring.
Reference


Thank you