The blocks of the periplectic Brauer algebra

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Schur-Weyl duality

\[ GL(m) \rightarrow (\mathbb{C}^m)^{\otimes n} \rightarrow \mathbb{C} \Sigma_n \]
Schur-Weyl duality

\[ \text{GL}(m) \bigcup \text{O}(m) \bigcap \mathbb{C} \Sigma_n \bigcap B_n(m) \cong (\mathbb{C}^m)^\otimes n \]
Schur-Weyl duality

$GL(2m) \cup \Sigma_C \cap Sp(2m) \rightarrow (\mathbb{C}^{2m})^\otimes n \leftarrow \mathbb{C}^\Sigma \cap B_n(-2m)$
Schur-Weyl duality

\[ \mathfrak{gl}(m|2k) \cup \mathfrak{osp}(m|2k) \subseteq (\mathbb{C}^{m|2k})^\otimes_n \cap B_n(m-2k) \]

\[ \mathbb{C}\Sigma_n \]
Schur-Weyl duality

\[ \mathfrak{gl}(m|m) \quad \cup \quad \mathfrak{pe}(m) \quad \Rightarrow \quad (\mathbb{C}^{m|m}) \otimes n \quad \Leftarrow \quad \mathbb{C}\Sigma_n \quad \cap \quad A_n \]
$(n, n)$-Brauer diagrams

$n$-northern nodes
(n, n)-Brauer diagrams

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

n-southern nodes
(n, n)-Brauer diagrams

Propagating lines
$(n, n)$-Brauer diagrams

Propagating lines, cups
$(n,n)$-Brauer diagrams

Propagating lines, cups and caps
$(n, n)$-Brauer diagrams

Only propagating lines $\Rightarrow$ Symmetric group
The (periplectic) Brauer algebra

To multiply two Brauer diagrams:
The (periplectic) Brauer algebra

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The (periplectic) Brauer algebra

To multiply two Brauer diagrams:

\[ \circ \]
\[ \delta \]

Replace each closed loop by \( \delta \),
\[ \delta = 0 \] for the periplectic case
The (periplectic) Brauer algebra

To multiply two Brauer diagrams:

Calculate the appropriate sign using certain rules.
Theorem (Kujawa–Tharp 2017)

The Brauer algebra $B_n(\delta)$, with $\delta \neq 0$:

- The $p$-restricted partitions of $n, n - 2, n - 4, \ldots, 0$ (n even)
- The $p$-restricted partitions of $n, n - 2, n - 4, \ldots, 1$ (n odd)

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is $p$-restricted if $\lambda_i - \lambda_{i+1} < p$ for all $i$. 
Labelling of simple modules

Theorem (Kujawa–Tharp 2017)

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Theorem (Kujawa–Tharp 2017)

The Brauer algebra $B_n(0)$:

- The $p$-restricted partitions of $n$, $n - 2$, $n - 4$, $\ldots$, $2$ (n even)
- The $p$-restricted partitions of $n$, $n - 2$, $n - 4$, $\ldots$, $1$ (n odd)

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is $p$-restricted if $\lambda_i - \lambda_{i+1} < p$ for all $i$. 
Labelling of simple modules

Theorem (Kujawa–Tharp 2017)

The periplectic Brauer algebra \( A_n \):

- The \( p \)-restricted partitions of \( n, n - 2, n - 4, \ldots, 2 \) (\( n \) even)
- The \( p \)-restricted partitions of \( n, n - 2, n - 4, \ldots, 1 \) (\( n \) odd)

A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is \( p \)-restricted if \( \lambda_i - \lambda_{i+1} < p \) for all \( i \).
$\lambda \sim \mu$ if there is a sequence

$$\lambda = \lambda_1, \lambda_2, \ldots, \lambda_t = \mu$$

with corresponding indecomposable $A$-modules

$$M_1, M_2, \ldots M_{t-1}$$

where $L(\lambda_i)$ and $L(\lambda_{i+1})$ appear as composition factors of $M_i$.

Each equivalence class corresponds to a block of $A$. 
Blocks

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Each equivalence class corresponds to a block of $A$. 
Decomposition of $A$ in indecomposable two-sided ideals:

$$A = B_1 \oplus B_2 \oplus \cdots \oplus B_k,$$

each $B_i$ is a block of $A$.

These blocks correspond also to a decomposition of the category of finite dimensional $A$-modules.
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Blocks of the periplectic Brauer algebra

**Theorem (Coulembier 2018)**

In characteristic zero, two partitions belong to the same block iff they have the same 2-core.

The 2-core: obtained by removing rim 2-hooks:

\[
\begin{align*}
\rho_0 &= \emptyset, & \rho_1 &= \begin{array}{c}
\hline
\end{array}, & \rho_2 &= \begin{array}{c}
\hline
\hline
\hline
\end{array}, & \rho_3 &= \begin{array}{c}
\hline
\hline
\hline
\hline
\hline
\end{array}, & \cdots
\end{align*}
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\[
\begin{array}{c}
\square \\
\end{array}
\text{ or }
\begin{array}{c}
\square \\
\end{array}
\]

The possible 2-core:

\[
\begin{array}{c}
\rho_0 = \emptyset, \\
\rho_1 = \begin{array}{c}
\square \\
\end{array}, \\
\rho_2 = \begin{array}{c}
\square \\
\square \\
\end{array}, \\
\rho_3 = \begin{array}{c}
\square \\
\square \\
\square \\
\end{array}, \\
\vdots
\end{array}
\]
**Blocks of the periplectic Brauer algebra**

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\hline
\hline
\end{array}
\end{array}, \quad \ldots
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Blocks in characteristic $p$

**Proposition**

*If two partitions have the same 2-core, they belong to the same block.*

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*If two partitions of equal size have the same $p$-core, they belong to the same block.*
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**Blocks in characteristic $p$**

**Proposition**

Consider the $r$-staircase partition $\rho_r$ with

$$2r - 1 < p \quad \text{and} \quad \frac{r(r + 1)}{2} + p - 2r > n.$$  

Then $\lambda \sim \rho_r$ if and only if the 2-core of $\lambda$ is $\rho_r$.

**Proposition**

If $\lambda$ has as 2-core $\rho_r$ not satisfying these conditions, then

- $\lambda \sim \emptyset$ (n even),
- $\lambda \sim \square$ (n odd).
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**Proposition**

If $\lambda$ has as 2-core $\rho_r$ not satisfying these conditions, then

- $\lambda \sim \emptyset$  \hspace{1cm} (n even),
- $\lambda \sim \Box$  \hspace{1cm} (n odd).
**Blocks in characteristic $p$**

**Theorem**

The block decomposition of $A_n$ is given by

$$B_n(\kappa) \oplus \bigoplus_{r} B_n(\rho_r).$$

Here $\kappa = (\square)$ if $n$ is odd or $\kappa = \emptyset$ if $n$ is even.

The sum is over all $r \geq 2$ such that

- $2r - 1 < p$,
- $\frac{r(r+1)}{2} + p - 2r > n$,
- $\frac{r(r+1)}{2} = n - 2k$.

In particular if $n \geq (p^2 + 7)/8$, there is only one block.
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