LOCAL PROPERTIES OF CUT GROUPS

Andreas Bächle
(joint with M. Caicedo, E. Jespers and S. Maheshwary)

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Theorem

\[
\lim_{n \to \infty} \frac{\ln c(p^n)}{\ln f(p^n)} = 1, \quad \text{for } p \in \{2, 3\}.
\]
Percentage of rational and cut groups in all groups

Rational groups vs Cut groups

Rational Cut
An old conjecture:

<table>
<thead>
<tr>
<th>Conjecture</th>
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**Conjecture**

\[ G \text{ rational}, \ P \in Syl_2(G) \implies P \text{ rational?} \]

In 2012, I.M. Isaacs and G. Navarro presented counterexamples of order 1536 to this conjecture. Yet, they also proved:

**Theorem (Isaacs-Navarro, 2012)**

Let \( G \) be a solvable rational group and \( P \in Syl_2(G) \) has nilpotency class at most 2. Then \( P \) is rational.
Problem A

\[ G \text{ cut, } P \in \text{Syl}_3(G) \implies P \text{ cut?} \]
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An element $x \in G$ is called inverse semi-rational (isr) in $G$ iff for all $y \in G$ s.t. $\langle x \rangle = \langle y \rangle$: $x \sim y$ or $x \sim y^{-1}$.

Hence:

$H$ cut $\iff \forall x \in H: x$ inverse semi-rational in $H$. 
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Hence:

$H \text{ cut } \iff \forall x \in H: x \text{ inverse semi-rational in } H.$

Lemma

Let $x \in G$ be a 3-element. Then

$x \text{ is isr in } G \iff x \text{ is isr in } P \text{ for some } P \in \text{Syl}_3(G) \text{ with } x \in P.$
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$G$ cut, $P \in \text{Syl}_3(G) \implies P$ cut?

Theorem

Let $G$ be a cut group and $P \in \text{Syl}_3(G)$. Then $P$ is also cut, provided one of the following holds:

1. $G$ is supersolvable,
2. $G$ is a Frobenius group,
3. $G$ is simple,
4. $G$ is of odd order and $O_3(G)$ is abelian,
5. $|G| \leq 2 \cdot 3^6$ or $|G| \in \{2 \cdot 3^6, 2 \cdot 3 \cdot 7^2\}$. 
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G cut, $P \in \text{Syl}_3(G)$ $\implies$ $P$ cut?

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4. $G$ is of odd order and $O_3(G)$ is abelian,
5. $|G| \leq 2000$ or $|G| \in \{2^2 \cdot 3^6, 2^3 \cdot 3^6, 2^2 \cdot 3^7\}$. 

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**Theorem (Chillag-Dolfi, 2010; B, 2017)**

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$G$ solvable. If $G$ is rational, then $\pi(G) \subseteq \{2, 3, 5\}$.
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**Theorem (Hegedűs, 2005)**

*If $G$ is a solvable rational group and $P \in \text{Syl}_5(G)$.
Then $P \trianglelefteq G$ and $\exp P | 5$.***
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**Theorem (Hegedűs, 2005)**

*If G is a solvable rational group and $P \in Syl_5(G)$, then $P \trianglelefteq G$ and $\exp P \mid 5$.***

**Remark**

*Let G be a solvable cut group and $p \in \{2, 3, 5, 7\}$, $P \in Syl_p(G)$. The $p$-length of G and the exponent of P can be arbitrarily large.*
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Let G be a solvable cut group and $p \in \{2, 3, 5, 7\}$, $P \in \text{Syl}_p(G)$. The $p$-length of G and the exponent of P can be arbitrarily large.

**Problem B**

Let G be a solvable cut group. Is it true that $\exp O_5(G) \mid 5$ and $\exp O_7(G) \mid 7$?
REFERENCES


A. Bächle, M. Caicedo, E. Jespers, S. Maheshwary, Global and local properties of finite groups with only finitely many central units in their integral group ring, 12 pages, submitted, 1808.03546 [math.GR], 2018.


Thank you for your attention!