Classes of pretopological spaces closed under the formation of final structures

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Abstract

We investigate coreflective subconstructs of the construct $Prtop$ of pretopological spaces and continuous maps and in particular the inclusion “order” between these subconstructs. We describe the smallest, second and third coreflective class and then all minimal elements that are strictly larger. Using these minimal elements we obtain a “partition” of the whole conglomerate of coreflective subconstructs of $Prtop$. The results dealing with classes in one member of this partition have an immediate interpretation in the framework of reflexive relations.

Keywords: Pretopological space; Final structure; Coreflective subconstruct; Inclusion order; Finitely generated space

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1. Introduction

The creation of pretopologies goes back to the beginning of the century. In this period, when topology was born, also more general structures were introduced. In 1907 Riesz [22] formulated an axiomatization of derived sets, more general than those for topological spaces. In 1935 Hausdorff studied so-called “Gestufte Räume” [8], linked to the sequential convergence described in L-spaces. In Čech’s book “Topological Spaces” [4], pretopological spaces (under the equivalent name of Čech closure spaces) form the framework for the study of topology.

In 1948 Choquet [5], later followed by Kowalsky [18], Fischer [7] and Kent [14], presented an axiomatic description of convergence theory based on filter convergence. In

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all these pioneering papers, in which the categories of convergence theory were founded, large attention was attributed to pretopological spaces, i.e., those convergence spaces in which convergence can be described by means of neighborhood filters (or equivalently by means of Čech closures). This attention comes from the fact that the pretopological reflection of a convergence space is a key instrument in convergence theory, in particular in finding appropriate extensions of topological notions such as denseness and regularity.

In the topological construct $Prtop$ of pretopological spaces and continuous maps final structures are formed in a very easy and elegant way. Moreover quotients in $Prtop$ are hereditary [3,12]. These nice features of $Prtop$ explain why recently pretopological spaces have proved to be useful in topology too. The construct $Prtop$ plays a key role in the classification of topological quotient maps since $Prtop$ is the extensional topological hull of $Top$ [2,3,11,13,16,25,26].

As the importance of pretopological spaces became clear, several authors investigated the construct $Prtop$ and some of its subconstructs [15]. Results on Cartesian closedness and exponential objects were obtained by Schwarz in [23,24] and by the authors in [19,20].

In this paper we consider those subconstructs of $Prtop$ that are closed under the formation of final sinks in $Prtop$, i.e., coreflective subconstructs of $Prtop$. These coreflective subconstructs arise naturally from so-called "limit operators in $Prtop"", which are analogous to the topological limit operators as studied by Herrlich in [9,10]. For instance if we assign to each subset $A$ of a pretopological space $X$ a subset $l_X A$ of all limits of $X$-converging sequences in $A$ then $l_X$ defines a limit operator in $Prtop$ and the subconstruct of $Prtop$ consisting of all pretopological spaces $X$ for which the Čech closure $cl_X$ coincides with $l_X$ is the coreflective subconstruct of all Fréchet spaces [17,19]. Using the same procedure starting with $l_X A$ the set of all points in the $X$-closure of some singleton subset of $A$, the associated coreflective subconstruct of $Prtop$ consists of all finitely generated spaces and is denoted by $Fing$ [19].

We investigate the inclusion "order" on the conglomerate of all coreflective subconstructs. We describe the smallest, second and third coreflective class and then all minimal elements strictly larger than these three. Using the minimal elements we obtain a partition of the whole conglomerate. One member of the partition consists of the coreflective subconstructs of $Fing$. Since $Fing$ is isomorphic to the construct $Relex$ of reflexive relations (or equivalently of spatial directed graphs) our results on the induced partial order have an immediate interpretation in the framework of reflexive relations.

The following notational conventions will be adopted.

When $X$ is a set and $A \subseteq \mathcal{P}(X)$, then

$$stack A = \{ E \subseteq X; \ A \subseteq E \text{ for some } A \in A \}.$$ 

When $x$ is an element of a set $X$, we shall denote $\hat{x}$ for the ultrafilter stack $\{ \{x\}\}$. If $f : X \to Y$ is a map between sets, and $\mathcal{F}$ is a filter on $X$, then its image by $f$ on $Y$ is

$$f(\mathcal{F}) = stack \{ f(F); \ F \in \mathcal{F} \}.$$
Infinite cardinals are denoted by \( \omega_\alpha \) where \( \alpha \) is an ordinal. When \( A \) is a set, \( |A| \) denotes its cardinality.

A pretopological space is a structured set \((X, q)\), where the structure \( q \) is a function assigning a neighborhood filter \( V_q(x) \) to each point \( x \in X \), and \( V_q(x) \) satisfies the condition \( V_q(x) \subseteq \hat{x} \).

When no confusion can arise, we simply write \( X \) instead of \((X, q)\) and \( V(x) \) instead of \( V_q(x) \).

A function \( f : (X, p) \to (Y, q) \) between pretopological spaces is continuous if \( V_q(f(x)) \subseteq f(V_p(x)) \) for each \( x \in X \).

If \( A \) is a class of pretopological spaces, \((Y, p)\) is a pretopological space and \( (f : (X, q) \to (Y, p)) \) is the sink consisting of all continuous maps from \( A \)-objects to \((Y, p)\), then we call this sink the total sink from \( A \)-objects to \((Y, p)\).

The construct of pretopological spaces and continuous maps is denoted by \( \text{Prtop} \).

In any pretopological space \( X \) a Čech closure operator \( \text{cl} \) is defined by

\[
x \in \text{cl} A \iff \forall V \in V(x) : V \cap A \neq \emptyset
\]

and the other way around, a Čech closure operator \( \text{cl} \) on \( X \) determines a unique pretopology on \( X \). \( \text{Prtop} \) is isomorphic to the construct of Čech closure spaces.

\( \text{Prtop} \) is a well-fibred topological construct. So in \( \text{Prtop} \) initial and final structures exist for arbitrary sources and sinks. We will make extensive use of final structures in \( \text{Prtop} \) and therefore recall their construction: if

\[
(f_i : (X_i, q_i) \to Y)_{i \in I}
\]

is a sink, then the final pretopological structure \( p \) on \( Y \) has as neighborhood filter in \( y \in Y \)

\[
V_p(y) = \hat{y} \quad \text{if} \quad y \notin \bigcup_{i \in I} f_i(X_i)
\]

and

\[
V_p(y) = \bigcap_{i \in I, x \in X_i, f_i(x) = y} f_i(V_{q_i}(x)) \quad \text{if} \quad y \in \bigcup_{i \in I} f_i(X_i).
\]

Categorical terminology follows that of Adámek, Herrlich and Strecker [1]. We will not use the abstract categorical theory since we only work in the setting of topological constructs.

All subconstructs are always assumed to be full and isomorphism-closed and to contain at least one nonempty space. Hence coreflective subconstructs are bireflective. We frequently use the following characterization of a coreflective subconstruct \( C \) of \( \text{Prtop} \): a subconstruct \( C \) of \( \text{Prtop} \) is coreflective if and only if the subconstruct \( C \) is closed under final \( \text{Prtop} \)-sinks, i.e., whenever \( (f_i : (X_i, q_i) \to (X, q)) \) is a final sink in \( \text{Prtop} \) and \((X_i, q_i)\) is a \( C \)-object for every \( i \) then \((X, q)\) is a \( C \)-object too. It is sufficient that \( C \) is closed under the formation of \( \text{Prtop} \)-coproducts and quotients.

If \( A \) is any class of pretopological spaces, then there exists a smallest coreflective subconstruct containing \( A \). It is denoted by \( \mathcal{H}(A) \) and if \( A \) is a singleton class \( \{A\} \),
2. The smallest coreflective subconstructs of Prtop

Subconstructs of Prtop are partially ordered by inclusion on their object-classes. Let \( \mathcal{P} \) be the conglomerate of all coreflective subconstructs of Prtop endowed with this partial order. In this section we describe the smallest, second and third smallest elements of \( \mathcal{P} \).

First we introduce some objects of Prtop that play an essential role in this investigation.

On the two-point set \( \{0, 1\} \) we will be dealing with \( 2 \), the topological Sierpinski space (i.e., \( \{\{0, 1\}, \{0\}, \{1\}, \{0, 1\}\} \)), and \( I_2 \), the topological indiscrete space.

On the three-point set \( \{0, 1, 2\} \) we will consider the pretopological space \( 3 \) with smallest neighborhoods \( \{0, 1, 2\} \) of \( 0 \) and \( 2 \) and \( \{1, 2\} \) of \( 1 \).

If \( \alpha \) is an ordinal number, \( X_\alpha \) is the set \( [0, \omega_\alpha] \), i.e.,

\[ \{\beta; \, \beta \text{ ordinal number, } 0 \leq \beta \leq \omega_\alpha\} \]

\( X_\alpha \) is the filter on \( X_\alpha \) generated by the sets \( \{0\} \cup [\beta, \omega_\alpha] \), with \( \beta < \omega_\alpha \); \( F(\alpha) \) is the pretopological space with \( X_\alpha \) as underlying set, with \( \{X_\alpha\} \) as neighborhood filter of each \( \beta < \omega_\alpha \), and \( X_\alpha \) as neighborhood filter of \( \omega_\alpha \).

Each of these objects \( X \) generates a coreflective hull \( \mathcal{H}(X) \). Some of these hulls have been studied earlier: the hull of \( 2 \) played an important role in the description of exponential objects in [19] and the hull of \( F(\alpha) \) played a key role in the investigation of Cartesian closedness for coreflective subconstructs of Prtop [20]. We recall two results from these earlier papers that will be useful in the sequel.

**Proposition 2.1** [19]. The coreflective hull of \( 2 \) in Prtop is the class Fing of all finitely generated spaces, i.e. all spaces \( X \) having the property that every point \( x \) has a smallest neighborhood \( V_x \).

**Proposition 2.2** [20]. (i) If \( X \) is a pretopological space, \( x \in X \) and \( \omega_\alpha \) is regular such that

1. \( V \subseteq V(x), \, |V| < \omega_\alpha \Rightarrow \bigcap V \in V(x) \),
2. \( \exists \{x; \, x \in \omega_\alpha \} \subseteq V(x); \, \bigcap_{x \in \omega_\alpha} V_x \notin V(x) \)

then there is a final sink \( (f: X \rightarrow F(\alpha)) \).

(ii) A coreflective subconstruct of Prtop that is not included in Fing, always contains a space \( F(\alpha) \), with \( \omega_\alpha \) a regular cardinal.

For the coreflective hulls of \( I_2 \) and \( 3 \) it is straightforward that the objects can be characterized in the following way.
Proposition 2.3. (i) The coreffective hull \( \mathcal{H}(I_2) \) consists of all finitely generated spaces \( X \) satisfying the symmetry condition
\[
a \in V_x \Rightarrow x \in V_a.
\]
(ii) The coreffective hull \( \mathcal{H}(3) \) consists of all finitely generated spaces \( X \) satisfying the condition
\[
a \in V_x \Rightarrow x \in V_a, \; z \in V_z, \; a \in V_z \text{ and } z \in V_a \text{ for some } z \in X.
\]

Theorem 2.4. (i) The class \( \text{Dis} \) consisting of all discrete spaces is the minimum element of \( \mathbb{P} \).
(ii) The coreffective hull \( \mathcal{H}(I_2) \) is the minimum element of \( \mathbb{P} \setminus \{\text{Dis}\} \).
(iii) The coreffective hull \( \mathcal{H}(3) \) is the minimum element of \( \mathbb{P} \setminus \{\text{Dis}, \mathcal{H}(I_2)\} \).

Proof. The result and proof of (i) and (ii) are quite similar to the topological situation studied by Herrlich in [9,10] and therefore we only give an explicit proof of (iii).

First remark that 3 does not satisfy the characterization of \( \mathcal{H}(I_2) \) of Proposition 2.3, so indeed \( \mathcal{H}(3) \) belongs to \( \mathbb{P} \setminus \{\text{Dis}, \mathcal{H}(I_2)\} \).

Let \( C \) be an arbitrary element of \( \mathbb{P} \setminus \{\text{Dis}, \mathcal{H}(I_2)\} \). First, we assume \( C \subset \text{Fing} \). Then \( C \) contains a finitely generated space \( X \) not satisfying the symmetry condition of Proposition 2.3(i). In this case let \( x, y \) in \( X \) such that \( x \in V_y, \; y \notin V_x \). If \( V_x = \{x\} \), then \( \varphi : X \to 2 \) mapping \( x \) to 1 and \( z \) to 0 if \( z \neq x \) is a quotient. This implies that \( C = \text{Fing} \) and so \( 3 \in C \). If \( V_x \neq \{x\} \), let \( \varphi_i : X \to 3 \) for \( i \in \{1, 2, 3\} \) be defined as follows:
- \( \varphi_1(x) = 1, \; \varphi_1(y) = 0, \; \varphi_1 \) equals 2 elsewhere,
- \( \varphi_2(x) = 0, \; \varphi_2 \) equals 2 elsewhere,
- \( \varphi_3(x) = 1, \; \varphi_3 \) equals 2 elsewhere.

Then \( \varphi_i : X \to 3, i \in \{1, 2, 3\} \) is a final sink and therefore \( 3 \in C \).

Secondly we assume that \( C \) is not included in \( \text{Fing} \). By Proposition 2.2(ii) there exists a regular cardinal \( \omega_\alpha \) such that \( F(\alpha) \) belongs to \( C \). Then consider the function \( \psi : F(\alpha) \to 3 \) defined by \( \psi(1) = 0, \; \psi(\omega_\alpha) = 1 \) and \( \psi \) equals 2 elsewhere. Then again \( \psi \) is a quotient and hence \( 3 \) belongs to \( C \). \( \Box \)

3. Coreffective subconstructs consisting of finitely generated spaces only

Let \( \mathbb{P} \) be the conglomerate of all coreffective subconstructs of \( \text{Fing} \) and consider the induced partial order. Of course, \( \text{Dis}, \mathcal{H}(I_2) \) and \( \mathcal{H}(3) \) are the smallest, second and third smallest elements of \( \mathbb{P} \). In order to make a further study of the partial order on \( \mathbb{P} \) it is useful to exploit the well-known isomorphism between the construct \( \text{Fing} \) of finitely generated pretopological spaces and the construct \( \text{Rere} \) of reflexive relations (also known as \( \text{SGraph} \), the construct of spatial graphs [6]). A space in \( \text{Rere} \) will be denoted by \( (X, \to) \) where \( \to \) is a reflexive relation. Morphisms between objects \( X \) and \( Y \) in \( \text{Rere} \) are set functions \( f : X \to Y \) with
\[
x \to x' \Rightarrow f(x) \to f(x')\]
A sink \((f_i : X_i \to X)_{i \in I}\) is final if and only if

\[
\exists k \in I, \exists x_k \in f_k^{-1}(x), \exists x_k' \in f_k^{-1}(x'):
\]

\[
x_k \leftrightarrow x_k' \iff x_k \leftrightarrow x_k' \text{ and } x' \in \bigcup_i f_i(X_i) \text{ or } x \neq x' \text{ and } x' \notin \bigcup_i f_i(X_i).
\]

If \(X\) is a Rere-object, then by describing the smallest neighborhood \(V_x\) of \(x\) as

\[V_x = \{x' \in X; x' \leftrightarrow x\}\]

a finitely generated pretopology on \(X\) corresponds. Leaving morphisms unchanged as set-functions, we get an isomorphism functor \(\text{Rere} \to \text{Fing}\). Thus, we can identify the constructs \(\text{Rere}\) and \(\text{Fing}\). So a finitely generated pretopological space can be represented as a diagram of arrows between its elements, where an arrow \(x \mapsto x'\) means that \(x\) belongs to the smallest neighborhood of \(x'\). An arrow \(x \leftrightarrow x'\) with \(x = x'\) is represented by a loop in \(x\), but is usually omitted in diagrams. If both the arrows \(x \mapsto x'\) and \(x' \mapsto x\) occur, we trace the double arrow \(x \leftrightarrow x'\). So for instance, using this representation the finitely generated space \(3\), we encountered earlier, is the following:

\[
\begin{array}{c}
1 \\
\downarrow \\
2
\end{array}
\]

where loops have to be added in each vertex.

To characterize the minimum of \(\mathbb{F} \setminus \{\text{Dis}, \mathcal{H}(I_2), \mathcal{H}(3)\}\) we introduce the finitely generated pretopological space \(6\) on the set \(\{0, 1, 2, 3, 4, 5\}\) picturized as follows:

\[
\begin{array}{c}
4 \\
\downarrow \\
3
\end{array}
\]

\[
\begin{array}{c}
0 \\
\downarrow \\
5
\end{array}
\]

\[
\begin{array}{c}
1 \\
\downarrow \\
2
\end{array}
\]

adding a double arrow between each pair of points that is not connected in the above diagram and a loop in each vertex.

The objects of the coreflective hull of \(6\) can easily be characterized as follows.
Proposition 3.1. The coreffective hull $\mathcal{H}(6)$ consists of all finitely generated spaces satisfying the following property: for each arrow $x_0 \rightarrow x_5$ in $X$ there exist points $x_1, x_2, x_3, x_4$ in $X$ such that we have the diagram

```
x_4   x_3
   ↓   ↓
x_0   x_5
   ↑   ↑
x_1   x_2
```

Adding a double arrow between each pair of points that is not connected in the above diagram and a loop in each vertex.

Theorem 3.2. The coreffective hull $\mathcal{H}(6)$ is the minimum of $\mathbb{F} \setminus \{Dis, \mathcal{H}(I_2), \mathcal{H}(3)\}$.

Proof. First remark that $6$ does not satisfy the characterization of the $\mathcal{H}(3)$-objects, given in Proposition 2.3(ii), hence $\mathcal{H}(6)$ belongs to $\mathbb{F} \setminus \{Dis, \mathcal{H}(I_2), \mathcal{H}(3)\}$.

Let $C$ be an arbitrary element of $\mathbb{F} \setminus \{Dis, \mathcal{H}(I_2), \mathcal{H}(3)\}$. Then $C$ contains an object $X$ which is a finitely generated space not belonging to $\mathcal{H}(3)$. Consider the total sink $X \rightarrow 6$, we prove that it is final. Let $g: 6 \rightarrow Z$ be a function and assume that $g \circ f$ is continuous whenever $f: X \rightarrow 6$ is continuous. Let $x \rightarrow x'$ be an arrow in $6$. If $(x, x') \neq (0, 5)$ then we can take $z \in 6$ such that $z$ and $x$ are as well as $z$ and $x'$ are connected by double arrows. Let $j: 3 \rightarrow 6$ be the continuous function mapping $0$ to $x'$, $1$ to $x$, and $2$ to $z$. Whenever $h: X \rightarrow 3$ is continuous we have that $g \circ j \circ h: X \rightarrow Z$ is continuous. Since the total sink from $X$ to $3$ is final by Theorem 2.4(iii), we can conclude that $g \circ j$ is continuous. It follows that $g(x) \rightarrow g(x')$.

Next suppose $(x, x') = (0, 5)$. Since $X$ does not belong to $\mathcal{H}(3)$ we can find $a$ and $a'$ in $X$ such that $a \leftrightarrow a'$ and no element $z$ in $X$ is connected to both $a$ and $a'$ by double arrows. Let $f: X \rightarrow 6$ be defined as follows: $f(a) = 0$ and $f(a') = 5$; if $z \in X$ is not connected to $a$ nor to $a'$ then we put $f(z) = 1$. In all other instances of $z \in X \setminus \{a, a\}$ at least one of the following cases will occur:

1. there is a single arrow $z \rightarrow a$,
2. there is a single arrow $a \rightarrow z$,
3. there is a single arrow $z \rightarrow a'$,
4. there is a single arrow $a' \rightarrow z$.

Put $f_1 = 3$, $f_2 = 4$, $f_3 = 1$, $f_4 = 2$ and define $f(z) = f_i$ if case $(i)$ occurs but none of the cases $(j)$ with $j < i$. The function $f: X \rightarrow 6$ thus defined is continuous and therefore $g \circ f$ is continuous too. It follows that $g(0) \leftrightarrow g(5)$. Finally we can conclude that $6$ belongs to $C$. $\Box$
Next we consider $\mathcal{F} \setminus \{F_{ng}\}$ and we investigate whether there is a maximum in this conglomerate. In order to do so, we introduce the following spaces: for $n \geq 1$, $Z_n$ is the finitely generated pretopological space on the set $\{0, 1, \ldots, n-1\}$ with the structure described by

![Diagram]

where the picture is completed by adding a loop in each vertex.

The proof of the following result is easy.

**Proposition 3.3.** (i) $f : Z_{n+1} \to Z_n$ mapping $n$ to 0 and $i$ identically to $i$ if $i \neq n$ is a quotient.

(ii) Every continuous function from $Z_n$ to $Z_{n+1}$ is constant.

It follows that $(\mathcal{H}(Z_n))_{n \geq 1}$ is a strictly increasing sequence in $\mathcal{F} \setminus \{F_{ng}\}$. Remark that $Z_1$ is the one-point discrete space generating the smallest element $\text{Dis}$ and $Z_2$ is the two-point indiscrete space generating the second element.

Next we consider $\{Z_n : n \geq 1\}$ and let $\mathcal{Z}$ be the coreflective hull $\mathcal{H}\{Z_n : n \geq 1\}$.

The proofs of the following results are straightforward.

**Proposition 3.4.** Let $X$ be finitely generated. The following are equivalent:

(i) $X$ belongs to $\mathcal{Z}$.

(ii) For every arrow $x \hookrightarrow x'$ there exists a finite sequence of points $x_1, x_2, \ldots, x_n$ with $x_n \hookrightarrow x$, $x' \hookrightarrow x_1$ and $x_j \hookrightarrow x_{j+1}$ for all $j \in \{1, 2, \ldots, n-1\}$.

(iii) The topological bireflection of $X$ is indiscretely generated, i.e., is a coproduct of indiscrete spaces.

**Theorem 3.5.** $\mathcal{Z}$ is the maximum element of $\mathcal{F} \setminus \{F_{ng}\}$. 
Proof. Let $C$ be a coreflective subconstruct of $Fing$, not included in $Z$. Then $C$ contains a space $X$ in which there is an arrow $x \rightarrow x_0$ such that for all finite sequences $x_1, x_2, \ldots, x_n$ of points in $X$ we don't have the diagram

\[
\begin{array}{ccc}
  x_0 & \rightarrow & x_1 \\
  \uparrow & & \downarrow \\
  x & \rightarrow & x_1 \\
  \downarrow & & \uparrow \\
  x_n & \rightarrow & x_2 \\
  \vdots & & \vdots \\
  & \rightarrow & \end{array}
\]

Define the subset

\[ A = \{x_n \in X; \exists x_1, x_2, \ldots, x_{n-1} \in X, \forall j \in \{0, 1, \ldots, n-1\}: x_j \leftrightarrow x_{j+1}\} \]

of $X$ and take the function $f : X \rightarrow \{0, 1\}$ with $f |_A = 0$ and $f |_{X \setminus A} = 1$. We show that $f : X \rightarrow 2$ is a quotient. It suffices to show that for a function $g : 2 \rightarrow Z$ such that $g \circ f$ is continuous, we have the arrow $g(1) \leftrightarrow g(0)$. Now $x \leftrightarrow x_0$ implies $g \circ f(x) \leftrightarrow g \circ f(x_0)$. But $f(x) = 1$ and $f(x_0) = 0$. □

Theorem 3.6. $\mathcal{F} \setminus \{Fing, Z\}$ has no maximal elements.

Proof. Let $C$ be an arbitrary element of $\mathcal{F} \setminus \{Fing, Z\}$. Since $C \neq Z$ there exists $k \geq 3$ such that $Z_k \notin C$. Put $\mathcal{D} = \mathcal{H}(C \cup \{Z_k\})$. Clearly $\mathcal{D}$ is strictly larger than $C$. So it suffices to prove that $\mathcal{D} \in \mathcal{F} \setminus \{Fing, Z\}$. Since all continuous maps from $Z_k$ to $Z_{k+1}$ are constant it follows that $Z_{k+1} \notin \mathcal{D}$. □

4. Coreflective subconstructs containing at least one nonlinearly generated space

In the study of coreflective subconstructs not contained in $Fing$, the objects $F(\alpha)$ introduced in Section 2 play an important role. We will also make use of the objects $C(\alpha)$ introduced in [9].

Let $\alpha$ be some ordinal and let $C(\alpha)$ be the (pre)topological space on the set $X_\alpha$ with the tails generating the neighborhood filter in $\omega_\alpha$ and all points $\beta < \omega_\alpha$ isolated. Then the coreflective hull of $C(\alpha)$ in $Prop$ consists of all $\omega_\alpha$-sequentially determined pretopological spaces, i.e., pretopological spaces in which the closure of a set can be derived from the convergence of an $\omega_\alpha$-net in that set and described in the introduction.
by means of a limit operator. In particular the coreflective hull of \(C(0)\) is the construct of all Fréchet pretopologies studied in [17,19]. Furthermore for an ordinal \(\alpha\), let \(\mathop{Prtop}_\alpha\) be the subconstruct of \(\mathop{Prtop}\) consisting of all pretopological spaces in which an intersection of strictly less than \(\omega_\alpha\) neighborhoods of a point again is a neighborhood of that point. It is easily seen that \(\mathop{Prtop}_\alpha\) is coreflective in \(\mathop{Prtop}\), that \(\mathop{Prtop}_0 = \mathop{Prtop}\), \(\mathop{Prtop}_\beta \subset \mathop{Prtop}_\alpha\) if \(\alpha \leq \beta\) and \(\cap_\alpha \mathop{Prtop}_\alpha = \mathop{Fin}\).

**Proposition 4.1.** Each pair of different objects from
\[
\{\mathcal{H}(6)\} \cup \{\mathcal{H}(F(\alpha)); \omega_\alpha \text{ regular}\}
\]
is incomparable.

**Proof.** Let \(\omega_\alpha\) and \(\omega_\beta\) be regular cardinals, \(\alpha < \beta\). Clearly \(F(\beta) \in \mathop{Prtop}_\beta\) and \(F(\alpha) \notin \mathop{Prtop}_\beta\). Hence \(\mathcal{H}(F(\alpha)) \notin \mathcal{H}(F(\beta))\).

Moreover \(F(\alpha)\) is \(\omega_\alpha\)-sequentially determined and \(F(\beta)\) is not \(\omega_\alpha\)-sequentially determined. Hence \(\mathcal{H}(F(\beta)) \notin \mathcal{H}(F(\alpha))\).

Clearly \(\mathcal{H}(F(\alpha)) \notin \mathcal{H}(6)\) since \(\mathcal{H}(6)\) consists of finitely generated spaces only.

Next we prove that \(6 \notin \mathcal{H}(F(\alpha))\). Suppose on the contrary that the total sink \((F(\alpha) \to 6)\) would be final. Then
\[
\text{stack}\{V_x\} = \bigcap_{f \text{ continuous}} f(V(f^{-1}(x))
\]
for every \(x \in 6\). Since \(0 \in V_5\) there exists a continuous function \(f : F(\alpha) \to 6\) such that \(\{0, 5\} \subset f(X_\alpha)\). Since \(5 \notin V_0\) we clearly have \(f^{-1}(0) = \{\omega_\alpha\}\). It follows that \(V(f^{-1}(5)) = \text{stack}\{X_\alpha\}\). Hence \(f(X_\alpha) \subset V_5\). Now choose \(\beta < \omega_\alpha\) such that \(f(\{0\} \cup [\beta, \omega_\alpha]) \subset V_6\). Let \(f(\beta) = b\), then \(b \neq 0\), \(b \neq 5\) and so \(f(X_\alpha) \subset V_6\). A contradiction follows. \(\Box\)

**Theorem 4.2.** \(\mathcal{H}(6)\) and \(\mathcal{H}(F(\alpha))\) for \(\omega_\alpha\) regular are minimal elements in
\[
P \setminus \{\mathop{Dis}, \mathcal{H}(I_2), \mathcal{H}(3)\}.
\]

**Proof.** That \(\mathcal{H}(6)\) is minimal follows from Theorem 3.2. If \(C\) belongs to
\[
P \setminus \{\mathop{Dis}, \mathcal{H}(I_2), \mathcal{H}(3)\}
\]
and \(C \subset \mathcal{H}(F(\alpha))\) with \(\omega_\alpha\) regular then \(C\) cannot be contained in \(\mathop{Fin}\) since otherwise \(6\) would belong to \(C\) and this would contradict the incomparability of \(\mathcal{H}(6)\) and \(\mathcal{H}(F(\alpha))\).

So by Proposition 2.2(ii) we have \(F(\beta) \in C\) for some regular cardinal \(\omega_\beta\). Again applying Proposition 4.1 we can conclude that \(C = \mathcal{H}(F(\alpha))\). \(\Box\)

If \(\omega_\alpha\) is a regular cardinal let \(P_\alpha\) be the conglomerate of all coreflective subconstructs \(\mathcal{C}\) with
\[
\mathcal{H}(F(\alpha)) \subset \mathcal{C} \subset \mathop{Prtop}_\alpha.
\]
Theorem 4.3. $\mathbb{P} \setminus \mathcal{F}$ is the disjoint union of all $\mathbb{P}_\alpha$ with $\omega_\alpha$ regular.

Proof. That $\mathbb{P}_\alpha$ and $\mathbb{P}_\beta$ are disjoint for $\alpha < \beta$ follows from the fact that $F(\alpha) \notin \text{Prtop}_\beta$. Let $C$ be any coreflective subconstruct of $\text{Prtop}$ with $C \notin \mathcal{F}$. Let

$$\alpha = \min \{ \gamma ; \exists X \in C, \exists x \in X, \exists \mathcal{U} \in \mathcal{V}(x), |\mathcal{U}| = \omega_\gamma, \bigcap \mathcal{U} \notin \mathcal{V}(x) \}.$$
Then $\omega_\alpha$ is a regular cardinal and clearly $C \subseteq \text{Prtop}_\alpha$. Moreover Proposition 2.2(i) implies that $F(\alpha) \in C$. 

**Corollary 4.4.** For regular cardinals $\omega_\alpha$ and $\omega_\beta$ with $\alpha \neq \beta$

$$\mathcal{H}(F(\alpha)) \cap \mathcal{H}(F(\beta)) = \mathcal{H}(F(\alpha)) \cap \text{Fin} = \mathcal{H}(3).$$

5. Subconstructs strictly larger than Fin

Let $M$ be the conglomerate of coreflective subconstructs strictly finer than Fin and for $\omega_\alpha$ regular $M_\alpha$ the conglomerate of all coreflective subconstructs $C$ with

$$\mathcal{H}(F(\alpha) + 2) \subset C \subset \text{Prtop}_\alpha.$$

**Theorem 5.1.** (i) The subconstructs $\{\mathcal{H}(F(\alpha) + 2), \omega_\alpha \text{ regular}\}$ are pairwise incomparable.

(ii) $M$ is the disjoint union of all $M_\alpha$ with $\omega_\alpha$ regular.

**Proof.** (i) Since $\mathcal{H}(F(\alpha) + 2) \subset \mathcal{H}(C(\alpha)) \subset \text{Prtop}_\alpha$ for $\omega_\alpha$ regular, we can repeat the argument used in Proposition 4.1.

(ii) Immediately from Theorem 4.3. 

References


